THE DEGENERATE MULTI-PHASE STEFAN PROBLEM WITH GIBBS-THOMSON LAW

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Abstract. In this paper we generalize the degenerate two–phase Stefan problem (Mullins–Sekerka evolution) to multi–phase systems. We prove a conditional existence result for this evolution problem in the framework of geometric measure theory by using an implicit time discretization. In each time step we solve a variational problem for an energy functional that contains capillarity terms as well as bulk energy contributions.

1. Introduction

The Stefan problem describes transitions between two phases. These are melting and solidification at the interface between liquid and solid. In the original formulation the phase transition was assumed to take place at a fixed melting temperature. This assumption had the drawback that undercooling (or superheating) can lead to regions of undefined phase (mushy regions). Therefore, the Gibbs–Thomson law was introduced which takes surface tension effects into account. This means that the melting temperature is not constant but proportional to the mean curvature of the interface. Existence of weak solutions to the non–degenerate Stefan problem with Gibbs–Thomson law was shown in [L1]. We refer to the book of Meirmanov [M] for the general theory of the Stefan problem.

In many physical situations heat conduction takes place on a faster time scale than the evolution of the interface. Having this in mind several authors (see for example [MS, Gu, LSt]) consider the quasi–static (or degenerate) Stefan problem with an infinite fast heat diffusion in the bulk. Existence of classical solutions local in time was shown by Chen, Hong, Yi [CHY] and Escher, Simonett [ES]. In [LSt] a conditional existence result for global solutions of this problem has been established by using an implicit time discretization of the problem. The crucial difficulty was to assure the convergence of the time–discrete solutions in an available sense. For that a condition which excludes a loss of perimeter in the limit had to be set. In this paper the result of [LSt] is generalized to multi–phase systems. We also

refer to Luckhaus [L2] and Bronsard, Garcke, Stoth [BGS] for the mathematical treatment of multicomponent alloy systems.

1.1 The equations. The degenerate Stefan problem with Gibbs–Thomson law is described formally by the equations

$$
(1) \qquad \partial_t \mathcal{X} = \Delta u \quad \text{in } \Omega \,,
$$

$$
(2) \qquad H = -(u + f) \quad \text{on } \Gamma
$$

where $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is open and bounded, Γ denotes the interface between liquid and solid, H is the mean curvature of Γ, u is the temperature, and $\mathcal{X}$ is the characteristic function of the set occupied by the liquid. The function f represents outer sources and sinks.

Equation (1) is the quasi–static version of the diffusion equation in the Stefan problem, incorporating Laplace's equation in bulk and a jump condition across the interface. The Gibbs–Thomson law (2) can be interpreted as the Euler–Lagrange equation of the Gibbs–Thomson free energy

$$
G(\bar{\mathcal{X}}) = \int_\Omega \left\{ |\nabla \bar{\mathcal{X}}| - (u + f)\bar{\mathcal{X}} \right\}
$$

which is defined for all characteristic functions $\bar{\mathcal{X}} \in BV(\Omega)$ (see [L2]).

In the case of N phases, we define $\mathcal{X}_i$ as the characteristic function of phase i, the interface between phases i and j is denoted by Γ_{ij}, and H_{ij} is its mean curvature. The specific energy content of phase i is given by the constant β_i and the surface tension on Γ_{ij} is a constant $\alpha_{ij} \in \mathbb{R}^+$. In this notation the energy diffusion equation for the N-phase system is given as

$$
(3) \qquad \sum_{i=1}^N \beta_i \partial_t \mathcal{X}_i = \Delta u \quad \text{in} \quad \Omega \times (0, T) \,.
$$

The Gibbs–Thomson law now demands that at all times t the characteristic functions $(\mathcal{X}_1(t), ..., \mathcal{X}_N(t))$ are a stationary point of

$$
G(\bar{\mathcal{X}}_1, ..., \bar{\mathcal{X}}_N) = \int_\Omega \left\{ \sum_{i,j=1, i<j}^N \alpha_{ij} \bar{\mu}_{ij} - \sum_{i=1}^N (\beta_i u + f_i) \bar{\mathcal{X}}_i \right\}
$$

in the class

$$
S := \left\{ (\mathcal{X}_1, \dots, \mathcal{X}_N) \in BV(\Omega, \{0,1\})^N, \sum_{i=1}^N \mathcal{X}_i = 1 \right\} \,.
$$

The measures μ_{ij} are the surface measures of the interface Γ_{ij} and they are defined by

$$
\mu_{ij} := \tfrac{1}{2} \left(|\nabla \mathcal{X}_i| + |\nabla \mathcal{X}_j| - |\nabla(\mathcal{X}_i + \mathcal{X}_j)| \right) \,.
$$

Hence, the Gibbs–Thomson relation becomes

$$
(4) \qquad \alpha_{ij} H_{ij} = (\beta_j - \beta_i)u + (f_j - f_i) \quad \text{on } \Gamma_{ij}.
$$

The functions f_i represent outer forces which result in generation and extermination of phase i.

A weak formulation for this problem is given in the setting of functions of bounded variation (see [Gi], [S]) as follows:

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and $\partial\Omega$ Lipschitz. Let $(\mathcal{X}_1^0, \ldots, \mathcal{X}_N^0) \in S$ and let $(f_1, \ldots, f_N) \in C(\overline{\Omega_T})^N$ with $\Omega_T := \Omega \times (0, T)$ for some fixed $T \in \mathbb{R}^+$ and let $u_D \in L^2(0, T; H^{1,2}(\Omega))$. Functions $(\mathcal{X}_1, \ldots, \mathcal{X}_N) \in L^\infty(0, T; S)$ together with a function $u \in u_D + L^2(0, T; \mathring{H}^{1,2}(\Omega))$ are called a weak solution of the degenerate multi–phase Stefan problem with Gibbs–Thomson law (3), (4) if

$$(5) \qquad \sum_{i=1}^N \beta_i \left(\int_{\Omega_T} \mathcal{X}_i \partial_t \xi + \int_\Omega \mathcal{X}_i^0 \xi(0) \right) = \int_{\Omega_T} \nabla u \nabla \xi,$$

for all $\xi \in C^\infty(\overline{\Omega_T}, \mathbb{R})$, $\xi(T) = 0$, $\xi|_{\partial\Omega \times (0,T)} = 0$

and

$$(6) \qquad \sum_{i,j=1, i<j}^N \int_{\Omega_T} \alpha_{ij} \left(\operatorname{div} \zeta - \frac{\nabla \mathcal{X}_i}{|\nabla \mathcal{X}_i|} \nabla \zeta \frac{\nabla \mathcal{X}_i}{|\nabla \mathcal{X}_i|} \right) \mu_{ij} + \sum_{i=1}^N \int_{\Omega_T} (\beta_i u + f_i) \zeta \nabla \mathcal{X}_i = 0,$$

for all $\zeta \in C^\infty(\overline{\Omega_T}, \mathbb{R}^n)$ with $\zeta \cdot \nu|_{\partial\Omega \times (0,T)} = 0$.

The orientation of Γ_{ij} is defined by the inner normal to the set occupied by the phase i. By ν we denote the outer normal to $\partial\Omega$ and we assume that the coefficients describing the specific energy content of the different phases $\beta_i \in \mathbb{R}$ are mutually different, i.e.

$$(7) \qquad \beta_i \neq \beta_j \quad \text{for } i \neq j .$$

The surface tension coefficients $\alpha_{ij} \in \mathbb{R}^+$ have to satisfy the subadditivity condition (see [L2])

$$(8) \qquad \alpha_{ij} \leq \alpha_{ik} + \alpha_{kj} \quad \text{for all } i, j, k \in \{1, \ldots, N\}$$

which will guarantee stability. We require symmetry, that is

$$(9) \qquad \alpha_{ij} = \alpha_{ji}$$

and we define $\alpha_{ii} := 0$.

The equation (6) is a weak formulation of (4) and it can be shown that (6) implies (4) for smooth surfaces. Furthermore, from (6) follows an angle condition at points where three interfaces meet (see [BGS]). To demonstrate that the measures μ_{ij} can be interpreted as surface measures on the interfaces Γ_{ij} we give a representation formula in terms of the reduced boundaries $\partial^*\Omega_i$ of the sets Ω_i (for a definition of the term reduced boundary see [Gi]). Then it holds for all open sets $\Omega' \subset \Omega$

$$(10) \qquad |\nabla \mathcal{X}_i|(\Omega') = \sum_{k=1, k \neq i}^N \mathcal{H}^{n-1} \left(\partial^*\Omega_i \cap \partial^*\Omega_k \cap \Omega' \right)$$

and

$$(11) \qquad |\nabla(\mathcal{X}_i + \mathcal{X}_j)|(\Omega') = \sum_{k=1, k \neq i,j}^N \left[\mathcal{H}^{n-1} \left(\partial^*\Omega_i \cap \partial^*\Omega_k \cap \Omega' \right) + \mathcal{H}^{n-1} \left(\partial^*\Omega_j \cap \partial^*\Omega_k \cap \Omega' \right) \right]$$

(for a proof see Vol'pert [V] and Baldo [B]). This implies

$$(12) \qquad \mu_{ij}(\Omega') = \mathcal{H}^{n-1}\left(\partial^*\Omega_i \cap \partial^*\Omega_j \cap \Omega'\right)$$

for all open sets $\Omega' \subset \Omega$. In addition, it holds

$$(13) \qquad \frac{\nabla\mathcal{X}_i}{|\nabla\mathcal{X}_i|} + \frac{\nabla\mathcal{X}_j}{|\nabla\mathcal{X}_j|} = 0 \qquad \mu_{ij} - \text{almost everywhere.}$$

1.2. The discretization. Similar to [LSt] we construct time-discrete solutions $(\mathcal{X}_1^h, \ldots, \mathcal{X}_N^h)$ and u^h for time steps $h > 0$. For that we have to solve the following minimum problem in each time step: minimize the functional

$$(14) \qquad
\begin{aligned}
\mathcal{F}^{h,t}(\bar{\mathcal{X}}_1, \ldots, \bar{\mathcal{X}}_N) &= \sum_{i,j=1, i<j}^{N} \alpha_{ij} \int_\Omega \bar\mu_{ij} + \frac{h}{2} \int_\Omega \nabla v \nabla(v - u_D^h) \\
&\quad - \sum_{i=1}^{N} \int_\Omega \left(f_i(t) + \beta_i u_D^h(t)\right) \bar{\mathcal{X}}_i \,,
\end{aligned}$$

in the class S, where the $\bar\mu_{ij}$ are the interface measures with respect to $(\bar{\mathcal{X}}_1, \ldots, \bar{\mathcal{X}}_N)$ and $v \in H^{1,2}(\Omega)$ is the weak solution of

$$(15) \qquad \sum_{i=1}^{N} \beta_i\left(\bar{\mathcal{X}}_i - \mathcal{X}_i^h(t-h)\right) = h\Delta v \,, \quad v = u_D^h(t)\big|_{\partial\Omega}$$

which is the implicit time discretization of (3) when $(\bar{\mathcal{X}}_1, \ldots, \bar{\mathcal{X}}_N) = (\mathcal{X}_1^h(t), \ldots, \mathcal{X}_N^h(t))$ and $v = u^h(t)$. The discretization of the boundary values u_D^h are chosen such that they are constant on time intervals $(kh, (k+1)h)$, $k \in \mathbb{N}$, and $u_D^h \to u_D$ in $L^2(0, T; H^{1,2}(\Omega))$ when h tends to zero. In addition we assume without loss of generality that the boundary values are extended into Ω such that $\Delta u_D(t) = \Delta u_D^h(t) = 0$ holds for almost all t. Minimizing the functional $\mathcal{F}^{h,t}$ is closely related to the equations (4) and (6) since the corresponding Euler-Lagrange equation is given by

$$(16) \qquad
\begin{aligned}
&\sum_{i,j=1, i<j}^{N} \int_\Omega \alpha_{ij}\left(\operatorname{div}\zeta - \frac{\nabla\mathcal{X}_i^h(t)}{|\nabla\mathcal{X}_i^h(t)|}\nabla\zeta\frac{\nabla\mathcal{X}_i^h(t)}{|\nabla\mathcal{X}_i^h(t)|}\right)\mu_{ij}^h \\
&\quad + \sum_{i=1}^{N} \int_\Omega (\beta_i u^h(t) + f_i)\zeta\nabla\mathcal{X}_i^h(t) = 0
\end{aligned}$$

$$\text{for all } \zeta \in C^\infty(\bar\Omega, \mathbb{R}^n) \qquad \text{with} \qquad \zeta \cdot \nu\big|_{\partial\Omega\times(0,T)} = 0\,.$$

This can be shown as follows. Let $\mathcal{X}^h(t)$ be a minimizer of $\mathcal{F}^{h,t}$ and let $u^h(t)$ be the solution of the elliptic boundary value problem (15) with $\bar{\mathcal{X}}_i = \mathcal{X}_i^h(t)$. Given a function ζ with properties as in (16) we define a family of transformations of the set $\bar\Omega$ by $\Psi : \bar\Omega \times \mathbb{R} \to \bar\Omega$

which we construct by solving the initial value problems

$$\frac{d}{d\varepsilon}\Psi(x,\varepsilon) = \zeta(\Psi(x,\varepsilon)),$$
$$\Psi(x,0) = x.$$

The condition for ζ on the boundary of Ω guarantee that $\Psi(\cdot,\varepsilon)$ maps $\bar\Omega$ in $\bar\Omega$. The transformations Ψ are constructed such that

$$\frac{d}{d\varepsilon}\Psi(x,\varepsilon)_{|\varepsilon=0} = \zeta(x),$$

i.e., $\Psi(\cdot,\varepsilon)$ transforms $\bar\Omega$ in the direction of the vector field ζ. Using the notation $\Omega_i^h := \{x \in \Omega \mid \mathcal{X}_i^h(t,x) = 1\}$ and $\Omega_{i,\varepsilon}^h := \{\zeta(x,\varepsilon) \mid x \in \Omega_i^h\}$ we define $\mathcal{X}_{i,\varepsilon}^h$ as the characteristic function of the set $\Omega_{i,\varepsilon}^h$ and v^ε as the corresponding solution of (15). The Euler–Lagrange equation (16) now follows from

$$(17) \qquad \frac{d}{d\varepsilon}\mathcal{F}^{h,t}(\mathcal{X}_{1,\varepsilon}^h, ..., \mathcal{X}_{N,\varepsilon}^h)_{|\varepsilon=0} = 0$$

which is true because $\mathcal{X}_{i,\varepsilon|\varepsilon=0}^h = \mathcal{X}_i^h(t)$ is the minimizer of (14). To compute the derivative with respect to ε of the first term in $\mathcal{F}^{h,t}(\mathcal{X}_{1,\varepsilon}^h, ..., \mathcal{X}_{N,\varepsilon}^h)$ (see (14)) one can apply the same arguments which are used to derive the first variation of area (see Giusti [Gi], Chapter 10). The non–standard part is to compute the derivative of

$$\frac{h}{2}\int_\Omega \nabla v^\varepsilon \nabla(v^\varepsilon - u_D^h) - \sum_{i=1}^N \int_\Omega \beta_i u_D^h \mathcal{X}_i^\varepsilon$$

with respect to ε. Defining $w^\varepsilon := v^\varepsilon - u_D^h$, $w := u^h(t) - u_D^h$ and using $\Delta u_D^h = 0$ we calculate

$$\begin{aligned}
\frac{h}{2}&\int_\Omega \nabla v^\varepsilon \nabla(v^\varepsilon - u_D^h) - \sum_{i=1}^N \int_\Omega \beta_i u_D^h \mathcal{X}_i^\varepsilon \\
&= \frac{h}{2}\int_\Omega |\nabla w^\varepsilon|^2 - \sum_{i=1}^N \int_\Omega \beta_i u_D^h \mathcal{X}_i^\varepsilon \\
&= \frac{h}{2}\int_\Omega |\nabla(w^\varepsilon - w)|^2 + h\int_\Omega \nabla(w^\varepsilon - w)\nabla w + \frac{h}{2}\int_\Omega |\nabla w|^2 - \sum_{i=1}^N \int_\Omega \beta_i u_D^h \mathcal{X}_i^\varepsilon \\
&= \frac{h}{2}\int_\Omega |\nabla(w^\varepsilon - w)|^2 - \sum_{i=1}^N \int_\Omega \beta_i(\mathcal{X}_{i,\varepsilon}^h - \mathcal{X}_i^h)w + \frac{h}{2}\int_\Omega |\nabla w|^2 - \sum_{i=1}^N \int_\Omega \beta_i u_D^h \mathcal{X}_i^\varepsilon \\
&= \frac{h}{2}\int_\Omega |\nabla(w^\varepsilon - w)|^2 - \sum_{i=1}^N \int_\Omega \beta_i \mathcal{X}_{i,\varepsilon}^h u^h(t) + \sum_{i=1}^N \int_\Omega \beta_i \mathcal{X}_i^h w + \frac{h}{2}\int_\Omega |\nabla w|^2
\end{aligned}$$

We observe that the first term is $o(\varepsilon)$ and that the third and fourth term do not depend on ε. Differentiating the second term by using the change of variables formula yields

$$\frac{d}{d\varepsilon}\left(\frac{h}{2}\int_\Omega \nabla v^\varepsilon \nabla(v^\varepsilon - u_D^h) - \sum_{i=1}^N \int_\Omega \beta_i u_D^h \mathcal{X}_i^\varepsilon\right)_{|\varepsilon=0} = \sum_{i=1}^N \int_\Omega \beta_i u^h(t)\zeta\nabla\mathcal{X}_i.$$

The derivative of the f_i–term can also be computed by using the change of variables formula. This proves that (16) is the Euler–Lagrange equation of the functional $\mathcal{F}^{h,t}$.

Starting with $\mathcal{X}_i^0$ one constructs $\mathcal{X}_i^h(kh)$, $k \in I\!N$, as the minimizer of $\mathcal{F}^{h,t}$ by iteration. For $t \in (kh, (k+1)h)$ we set $\mathcal{X}_i^h(t) = \mathcal{X}_i^h((k+1)h)$. The existence of a minimizer of $\mathcal{F}^{h,t}$ can be seen as follows: From the BV–compactness–theorem we immediately get the convergence of a minimizing sequence $(\bar{\mathcal{X}}_1^l, \ldots, \bar{\mathcal{X}}_N^l)_{l \in I\!N}$ in $L^2(\Omega)^N$. Defining v_l as the solution of (15) with $\bar{\mathcal{X}}_i = \bar{\mathcal{X}}_i^l$ we conclude that $v^l \to v$ in $H^{1,2}(\Omega)$ and we establish the continuity of the ∇v–term in the functional $\mathcal{F}^{h,t}$. To finish the existence proof we need a lemma which assures the lower semicontinuity of the interfacial energy term in $\mathcal{F}^{h,t}$.

Lemma 1.1. *Let $\alpha_{ij} \in I\!R^+$ $(i,j = 1, \ldots, N)$ satisfy (8). Then the functional*

$$\mathcal{A}(\mathcal{X}_1, \ldots, \mathcal{X}_N) = \sum_{i,j=1, i<j}^N \alpha_{ij} \int_\Omega \mu_{ij}$$

is lower semicontinuous in the class S with respect to strong convergence in $L^1(\Omega)$.

Proof. Assume $(\mathcal{X}_1^l, \ldots, \mathcal{X}_N^l) \to (\mathcal{X}_1, \ldots, \mathcal{X}_N)$ in $L^1(\Omega)$. We define measures μ_i^l by

$$\mu_i^l(D) := \int_D \Big| \nabla \sum_{j=1}^N \alpha_{ij} \mathcal{X}_j^l \Big|, \qquad i = 1, \ldots, N$$

for all $D \subset \Omega$ open. Baldo [B] demonstrated that under the assumption (8) the equality

$$(18) \qquad \bigvee_{i=1}^N \mu_i^l = \sum_{i,j=1, i<j}^N \alpha_{ij} \mu_{ij}^l$$

holds. By $\bigvee_{i=1}^N \mu_i^l$ we denote the measure theoretic supremum of the measures $\mu_1^l, \ldots, \mu_N^l$, which is defined by

$$\bigvee_{i=1}^N \mu_i^l(D) := \sup\Big\{ \sum_{i=1}^N \mu_i^l(D_i) \mid D_i \subset D \quad \text{open and mutually disjoint} \Big\}.$$

Let $D_i \subset \Omega$ be mutually disjoint and let $g_i \in C_0^1(D_i, I\!R^n)$ with $|g_i| \leq 1$. Then we may estimate

$$\sum_{i,j=1}^N \int_\Omega \alpha_{ij} \mathcal{X}_j \, \mathrm{div}\, g_i \; = \lim_{l \to \infty} \sum_{i,j=1}^N \int_\Omega \alpha_{ij} \mathcal{X}_j^l \, \mathrm{div}\, g_i$$

$$\leq \liminf_{l \to \infty} \bigvee_{i=1}^N \mu_i^l(\Omega).$$

Taking the supremum over all possible choices of D_i and g_i gives

$$\bigvee_{i=1}^N \mu_i(\Omega) \leq \liminf_{l \to \infty} \bigvee_{i=1}^N \mu_i^l(\Omega).$$

Applying the identity (18) gives the assertion of the lemma. $\qquad\qquad\square$

2. The existence result

We first prove the compactness of the discrete solutions $(\mathcal{X}_1^h, \ldots, \mathcal{X}_N^h)$ in $L^1(\Omega_T)$. This will be shown with the help of the following four lemmas. We assume that $\partial\Omega$ is Lipschitz. First we show an a priori estimate.

Lemma 2.1 (a priori estimate). *The discrete solutions $(\mathcal{X}_1^h, \ldots, \mathcal{X}_N^h)$ fulfill*

$$\sup_{t \in (0,T)} \sum_{i,j=1, i<j}^{N} \int_\Omega \mu_{ij}^h(t) + \int_{\Omega_T} |\nabla u^h|^2 \leq C$$

where C is a constant depending on $\int_\Omega |\nabla \mathcal{X}_i^0|$, $\|u_D\|_{L^2(0,T;H^{1,2}(\Omega))}$, $\|f_i\|_{L^1(\Omega_T)}$, α_{ij} and β_i $(i,j=1,\ldots,N)$.

Proof. We sum $\mathcal{F}^{h,t}(\mathcal{X}_1^h(kh), \ldots, \mathcal{X}_N^h(kh))$ over k. Then we use that $(\mathcal{X}_1^h(kh), \ldots, \mathcal{X}_N^h(kh))$ is a minimizer of $\mathcal{F}^{h,t}$. This implies in particular

$$\mathcal{F}^{h,t}(\mathcal{X}_1^h(kh), \ldots, \mathcal{X}_N^h(kh)) \leq \mathcal{F}^{h,t}(\mathcal{X}_1^h((k-1)h), \ldots, \mathcal{X}_N^h((k-1)h)).$$

Now we can establish the a priori estimate with the help of Young's inequality. $\square$

Lemma 2.2 (Compactness in space). *The discrete solutions $(\mathcal{X}_1^h, \ldots, \mathcal{X}_N^h)$ fulfill*

$$\int_{\Omega_{T'}} |\mathcal{X}_i^h(\cdot + s\vec{e}) - \mathcal{X}_i^h| \xrightarrow[s \to 0]{} 0$$

uniformly in h for each unit vector $\vec{e} \in \mathbb{R}^n$, each $1 \leq i \leq N$, and each $\Omega_{T'} \subset\subset \Omega_T$.

Proof. By approximation with smooth functions we can establish that

$$(19) \qquad \int_{\Omega_{T'}} |\mathcal{X}_i^h(\cdot + s\vec{e}) - \mathcal{X}_i^h| \leq s \int_{\Omega_T} |\nabla \mathcal{X}_i^h|$$

holds if s is sufficiently small. Since

$$\int_{\Omega_T} |\nabla \mathcal{X}_i^h| \leq \frac{1}{\min_{ij} \alpha_{ij}} \sum_{i,j=1, i<j}^{N} \alpha_{ij} \int_{\Omega_T} \mu_{ij}^h,$$

we have the uniform boundedness of $\int_{\Omega_T} |\nabla \mathcal{X}_i^h|$ from the a priori estimate in Lemma 2.1. Hence the claim follows from (19). $\square$

The next lemma is needed in order to control time differences of the solution.

Lemma 2.3. *Let $\varphi \in BV(\Omega)$, $\|\varphi\|_\infty \leq M \in \mathbb{R}^+$. Then there exist constants $c, \rho_0 \in \mathbb{R}^+$ depending on Ω and M such that for all $\rho < \rho_0$*

$$\int_\Omega |\varphi| \leq \rho \left(\int_\Omega |\nabla \varphi| + c \mathcal{H}^{n-1}(\partial\Omega) \right) + \frac{c}{\rho} \|\varphi\|_{H^{-1,2}(\Omega)}.$$

Proof. Let $\psi_\rho = \rho^{-n}\psi(\frac{x}{\rho})$ be a smooth mollifier with compact support. We split

$$\text{(20)} \qquad \int_\Omega |\varphi| \leq \int_\Omega |\varphi * \psi_\rho - \varphi| + \int_\Omega |\varphi * \psi_\rho|$$

where we extend φ by zero on $I\!\!R^n \setminus \Omega$. In order to treat the first term on the righthandside of (20) we consider an arbitrary $u \in C^\infty(\overline{\Omega})$, $|u| \leq M$. Choosing $\eta_\epsilon \in C_0^\infty(\Omega)$, $\eta_\epsilon \geq 0$, $|\nabla \eta_\epsilon| \leq c/\epsilon$, $\eta_\epsilon = 1$ in $\Omega_\epsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \epsilon\}$ we compute

$$
\begin{aligned}
\int_\Omega |(u\eta_\epsilon) * \psi_\rho - u\eta_\epsilon| &\leq \rho \int_\Omega |\nabla(u\eta_\epsilon)| \\
&\leq \rho \int_\Omega \eta_\epsilon |\nabla u| + c\frac{\rho}{\epsilon}\int_{\Omega \setminus \Omega_\epsilon} |u| \\
&\leq \rho\left(\int_\Omega |\nabla u| + c\mathcal{H}^{n-1}(\partial\Omega)\right)
\end{aligned}
$$

for $\rho \leq \rho_0$ with suitable ρ_0, c. Passing to the limit $\epsilon \to 0$ and approximating φ by smooth functions u (see [Gi]) we obtain

$$\int_\Omega |\varphi * \psi_\rho - \varphi| \leq \rho\left(\int_\Omega |\nabla\varphi| + c\mathcal{H}^{n-1}(\partial\Omega)\right) .$$

The second term in (20) is estimated by

$$\int_\Omega |\varphi * \psi_\rho| \leq \frac{c}{\rho}\|\varphi\|_{H^{-1,2}(\Omega)} .$$

This proves the lemma. $\qquad\square$

Lemma 2.4 (Compactness in time). *The discrete solutions* $(\mathcal{X}_1^h, \ldots, \mathcal{X}_N^h)$ *satisfy*

$$\int_0^{T-\tau}\int_\Omega \|\sum_{i=1}^N \beta_i\big(\mathcal{X}_i^h(\cdot + \tau) - \mathcal{X}_i^h\big)\| \leq c\tau^{1/4} .$$

Proof. We can assume $\tau = kh$, $t = mh$ for some $k, m \in I\!\!N$. For abbreviation we set $\varphi = \sum_{i=1}^N \beta_i(\mathcal{X}_i^h(\cdot + \tau) - \mathcal{X}_i^h)$. Using that $(\mathcal{X}_1^h, \ldots, \mathcal{X}_N^h)$ is a solution of the implicit time discretization of the diffusion equation and the a priori estimate (Lemma 2.1) we estimate

$$
\begin{aligned}
\|\varphi\|_{H^{-1,2}(\Omega)} &\leq \int_t^{t+\tau}\left\|\sum_{i=1}^N \beta_i \frac{\mathcal{X}_i^h(s) - \mathcal{X}_i^h(s-h)}{h}\right\|_{H^{-1,2}(\Omega)} \\
&\leq \tau^{1/2}\left(\int_t^{t+\tau} \|u^h(s)\|_{H^{1,2}(\Omega)}^2\right)^{\frac{1}{2}} \\
&\leq c\tau^{1/2} .
\end{aligned}
$$

Employing the last lemma with $\rho = \tau^{1/4}$ finishes the proof. $\qquad\square$

The results of the Lemmas 2.2 and 2.4 enable us to apply the Fréchet–Kolmogoroff compactness theorem to deduce the existence of a subsequence of $(\mathcal{X}_1^h, \ldots, \mathcal{X}_N^h)$ (for which we use the same notation as for the whole sequence) such that

$$(21) \qquad \sum_{i=1}^{N} \beta_i \mathcal{X}_i^h \xrightarrow[h \to 0]{} \mathcal{X} \quad \text{in } L^1(\Omega_T)$$

for some $\mathcal{X} \in L^1(\Omega_T)$. A subsequence converges in addition almost everywhere and therefore, $\mathcal{X}$ only attains the values β_i, $i = 1, \ldots, N$. Defining $\Omega_i := \{ x \mid \mathcal{X}(x) = \beta_i \}$ and $\mathcal{X}_i$ as the characteristic functions of Ω_i we conclude $\mathcal{X}_i^h \to \mathcal{X}_i$ almost everywhere and therefore

$$(22) \qquad \mathcal{X}_i^h \xrightarrow[h \to 0]{} \mathcal{X}_i \quad \text{in } L^1(\Omega_T) \quad \text{for all } 1 \le i \le N$$

with $(\mathcal{X}_1, \ldots, \mathcal{X}_N) \in L^\infty(0, T; S)$. From (22) and the a priori estimates of Lemma 2.1 we obtain that

$$(23) \qquad \nabla \mathcal{X}_i^h \rightharpoonup \nabla \mathcal{X}_i \quad \text{for all } 1 \le i \le N$$

in the sense of Radon-measures. Now we want to pass to the limit in the implicit time discretization of the diffusion equation and in the weak formulation of the Gibbs–Thomson law. To do so we have to require that there is no loss of perimeter in the limit, i.e.

$$(24) \qquad \int_{\Omega_T} |\nabla \mathcal{X}_i^h| \to \int_{\Omega_T} |\nabla \mathcal{X}_i| \quad \text{for all } 1 \le i \le N .$$

This condition gives us control on the limit behaviour of the normals $\nabla \mathcal{X}_i^h / |\nabla \mathcal{X}_i^h|$. Conditions of this form are typical in this type of geometric problems and we refer to [ATW, LSt, BGS, O] where the same conditions were assumed for other geometric problems.
Our main result is:

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and $\partial\Omega$ Lipschitz. Suppose $\beta_i \in \mathbb{R}$, $1 \le i \le N$, and $\alpha_{ij} \in \mathbb{R}^+$, $1 \le i, j \le N$, satisfy (7), (8) and (9) and assume that (24) holds. Then there exist functions $(\mathcal{X}_1, \ldots, \mathcal{X}_N) \in L^\infty(0, T; S)$ and $u \in u_D + L^2(0, T; \overset{\circ}{H}^{1,2}(\Omega))$ which are a weak solution of the degenerate multi-phase Stefan problem with Gibbs–Thomson law (3), (4)*

Proof. Since u^h is bounded in $L^2(0, T; H^{1,2}(\Omega))$ uniformly in h (see Lemma 2.1) we get the existence of a weakly convergent subsequence

$$u^h \rightharpoonup u \quad \text{in } L^2(0, T; H^{1,2}(\Omega)) .$$

From the weak completeness of $L^2(0, T; \overset{\circ}{H}^{1,2}(\Omega))$ we see that

$$u \in u_D + L^2(0, T; \overset{\circ}{H}^{1,2}(\Omega)) .$$

A discrete integration by parts of the term $\int_{\Omega_T} \partial_t^{-h} \mathcal{X}_i^h \xi$ gives (5) if we pass to the limit $h \to 0$ in the implicit time discretization of the diffusion equation (see (15) for $\bar{\mathcal{X}}_i = \mathcal{X}_i^h(t)$ and $v = u^h(t)$).

It remains to show (6), which we shall derive from (16) in the limit $h \to 0$. The convergence of the term $\int_\Omega (\beta_i u^h + f_i) \zeta \nabla \mathcal{X}_i^h$ follows since we can use the identity

$$\int_\Omega (\beta_i u^h + f_i) \zeta \nabla \mathcal{X}_i^h = - \int_\Omega \operatorname{div}(\beta_i u^h \zeta) \mathcal{X}_i^h + \int_\Omega f_i \zeta \nabla \mathcal{X}_i^h .$$

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With the convergence we established above it is possible to pass to the limit in the $\beta_i u$– and the f_i–terms.

For the treatment of the curvature term we first show

$$(25) \qquad \qquad \mu_{ij}^h \rightharpoonup \mu_{ij}$$

in the sense of Radon measures. We will always consider the measures as measures defined on $I\!\!R^n$ by extending them by zero outside of Ω. To establish (25) it is sufficient to show $\int_\Omega f \mu_{ij}^h \to \int_\Omega f \mu_{ij}$ for all $f \in C^0(\bar{\Omega})$.

Assumption (24) and the lower semicontinuity of $|\nabla \mathcal{X}_i^h|$ on open sets imply

$$\limsup_{h \to 0} \int_{K \cap \Omega} |\nabla \mathcal{X}_i^h| = \limsup_{h \to 0} \left(\int_\Omega |\nabla \mathcal{X}_i^h| - \int_{\Omega \setminus K} |\nabla \mathcal{X}_i^h| \right)$$
$$\leq \int_\Omega |\nabla \mathcal{X}_i| - \int_{\Omega \setminus K} |\nabla \mathcal{X}_i| = \int_{K \cap \Omega} |\nabla \mathcal{X}_i|$$

for all compact $K \subset I\!\!R^n$. Hence we can use a theorem which characterizes weak convergence of Radon measures by lower semicontinuity of the measures on open sets and upper semicontinuity on compact sets (see Theorem 1 in Section 1.9 of [EG]). This result implies

$$(26) \qquad \qquad \int_\Omega f |\nabla \mathcal{X}_i^h| \to \int_\Omega f |\nabla \mathcal{X}_i|$$

for all $f \in C^0(\bar{\Omega})$. From (10)–(12) we get the representation

$$\int_\Omega f |\nabla \mathcal{X}_i^h| = \sum_{j=1, j \neq i}^N f \mu_{ij}^h.$$

Using the lower semicontinuity

$$\liminf_{h \to 0} \int_{\Omega'} |\nabla (\mathcal{X}_i^h + \mathcal{X}_j^h)| \geq \int_{\Omega'} |\nabla (\mathcal{X}_i + \mathcal{X}_j)|$$

which holds for all open sets $\Omega' \subset \Omega$ then gives for all $f \in C^0(\bar{\Omega})$, $f \geq 0$

$$(27) \qquad \qquad \liminf_{h \to 0} \int_\Omega f |\nabla (\mathcal{X}_i^h + \mathcal{X}_j^h)| \geq \int_\Omega f |\nabla (\mathcal{X}_i + \mathcal{X}_j)|.$$

This follows as in the proof of Theorem 1 in Section 1.9 of [EG].

From (26) and (27) we conclude that for all $f \in C^0(\bar{\Omega})$, $f \geq 0$

$$(28) \qquad \qquad \limsup_{h \to 0} \int_\Omega f \mu_{ij}^h \leq \int_\Omega f \mu_{ij}$$

A strict inequality in (28) would imply that for nonnegative $f \in C^0(\bar{\Omega})$

$$
\begin{aligned}
\int_\Omega f |\nabla \mathcal{X}_i| &= \lim_{h \to 0} \int_\Omega f |\nabla \mathcal{X}_i^h| \\
&= \lim_{h \to 0} \int_\Omega \sum_{j=1, j \neq i}^N f \mu_{ij}^h \\
&\leq \sum_{j=1, j \neq i}^N \limsup_{h \to 0} \int_\Omega f \mu_{ij}^h \\
&< \sum_{j=1, j \neq i}^N \int_\Omega f \mu_{ij} = \int_\Omega f |\nabla \mathcal{X}_i|
\end{aligned}
$$

which is a contradiction. Since the above argument works for all subsequences we conclude

$$
\lim_{h \to 0} \int_\Omega f \mu_{ij}^h = \int_\Omega f \mu_{ij}
$$

for all nonnegative $f \in C^0(\bar{\Omega})$. A splitting of a general $f \in C^0(\bar{\Omega})$ in a positive and a negative part then gives (25).

Hence the convergence

$$
\sum_{i,j=1, i<j}^N \int_\Omega \alpha_{ij} \operatorname{div} \zeta \mu_{ij}^h \to \sum_{i,j=1, i<j}^N \int_\Omega \alpha_{ij} \operatorname{div} \zeta \mu_{ij}
$$

is established. It remains to show

$$
(29) \qquad \int_\Omega \nu_i^h \nabla \zeta \nu_i^h \mu_{ij}^h \to \int_\Omega \nu_i \nabla \zeta \nu_i \mu_{ij}
$$

where we set $\nu_i^h := \frac{\nabla \mathcal{X}_i^h}{|\nabla \mathcal{X}_i^h|}$ and $\nu_i := \frac{\nabla \mathcal{X}_i}{|\nabla \mathcal{X}_i|}$. Therefore, we need approximative normals $g_\varepsilon \in C_0^\infty(\Omega, \mathbb{R}^n)$, $|g_\varepsilon| \leq 1$ such that

$$
\int_\Omega (1 - g_\varepsilon \nu_i) |\nabla \mathcal{X}_i| \leq \varepsilon.
$$

The existence of such g_ε follows from the definition of $\int_\Omega |\nabla \mathcal{X}_i|$. From (23) and (24) we get

$$
\begin{aligned}
(30) \qquad \lim_{h \to 0} \int_\Omega (1 - g_\varepsilon \nu_i^h) |\nabla \mathcal{X}_i^h| &= \lim_{h \to 0} \int_\Omega \left(|\nabla \mathcal{X}_i^h| - g_\varepsilon \nabla \mathcal{X}_i^h \right) \\
&= \int_\Omega |\nabla \mathcal{X}_i| - \int_\Omega g_\varepsilon \nabla \mathcal{X}_i \\
&= \int_\Omega (1 - g_\varepsilon \nu_i) |\nabla \mathcal{X}_i| \leq \varepsilon
\end{aligned}
$$

which implies that g_ε is also a good approximation of the normals ν_i^h if h is small. Now we estimate with a constant C depending on ζ:

$$\left| \int_\Omega \nu_i^h \nabla \zeta \nu_i^h \mu_{ij}^h - \int_\Omega \nu_i \nabla \zeta \nu_i \mu_{ij} \right| =$$

$$= \left| \int_\Omega (\nu_i^h - g_\varepsilon) \nabla \zeta (\nu_i^h + g_\varepsilon) \mu_{ij}^h + \int_\Omega g_\varepsilon \nabla \zeta g_\varepsilon (\mu_{ij}^h - \mu_{ij}) + \int_\Omega (g_\varepsilon - \nu_i) \nabla \zeta (\nu_i + g_\varepsilon) \mu_{ij} \right|$$

$$\leq C \int_\Omega |\nu_i^h - g_\varepsilon| \, |\nabla \mathcal{X}_i^h| + \left| \int_\Omega g_\varepsilon \nabla \zeta g_\varepsilon (\mu_{ij}^h - \mu_{ij}) \right| + C \int_\Omega |\nu_i - g_\varepsilon| \, |\nabla \mathcal{X}_i|$$

Therefore, it follows

(31)

$$\limsup_{h \to 0} \left| \int_\Omega \nu_i^h \nabla \zeta \nu_i^h \mu_{ij}^h - \int_\Omega \nu_i \nabla \zeta \nu_i \mu_{ij} \right| \leq$$

$$\leq C \left(\limsup_{h \to 0} \int_\Omega |\nu_i^h - g_\varepsilon| \, |\nabla \mathcal{X}_i^h| + \int_\Omega |\nu_i - g_\varepsilon| \, |\nabla \mathcal{X}_i| \right)$$

Since

$$\left| \nu_i^h - g_\varepsilon \right|^2 = |\nu_i^h|^2 - 2 g_\varepsilon \nu_i^h + |g_\varepsilon|^2 \leq 2 \left(1 - g_\varepsilon \nu_i^h \right)$$

the convergence (29) follows from (30) and (31) for $\varepsilon \to 0$. This proves the theorem. $\qquad \square$

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