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THE THIN VISCOUS FLOW EQUATION IN HIGHER SPACE DIMENSIONS

Michiel Bertsch, Roberta Dal Passo
Dipartimento di Mathematica, Università di Tor Vergata
Via della Ricerca Scientifica, 00133 Roma, Italy

Harald Garcke
Institut für Angewandte Mathematik, Wegelerstr. 6, 53115 Bonn, Germany

Günther Grün
Institut für Angewandte Mathematik, Beringstr. 6, 53115 Bonn, Germany

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Abstract. We prove local integral (entropy) estimates for nonnegative solutions of the fourth-order degenerate parabolic equation

$$u_t + \operatorname{div}(u^n \nabla \Delta u) = 0$$

in space dimensions two and three. These estimates enable us to show that solutions have finite speed of propagation if $n \in (\frac{1}{8}, 2)$ and that the support cannot shrink if the growth exponent n is larger than 3/2. In addition, we prove decay estimates for solutions of the Cauchy problem and a growth estimate for their support.

1. Introduction. We consider nonnegative solutions of the initial boundary value problem

$$\begin{cases} u_t + \operatorname{div}\big(u^n \nabla \Delta u\big) = 0 & \text{in } \Omega \times (0,T) \\ \frac{\partial}{\partial \nu} u = \frac{\partial}{\partial \nu} \Delta u = 0 & \text{on } \partial\Omega \times [0,T] \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases} \tag{P_T}$$

in space dimensions two and three. This problem appears in the lubrication theory for thin viscous films that are driven by surface tension and the function u is the height of the film (cf. [15]). The above partial differential equation is a fourth-order parabolic equation that degenerates for $u = 0$. In recent years also other examples of similar degenerate parabolic equations of higher order appeared in the physics and materials science literature. We

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only mention models for phase separation in alloys (Cahn–Hilliard equation with degenerate mobility; cf. [8], [10], [11]) and models for the evolution of dislocation densities in the theory of plasticity (Norton–Hoff-type models; cf. [14]). For an overview on degenerate parabolic equations of higher order and their applications we refer to Bernis [2].

The mathematical investigation of problem (P_T) started with a paper by Bernis and Friedman ([5]). In one space dimension they were able to show the existence of a nonnegative Hölder-continuous solution for all values $n \geq 1$. The Hölder continuity of the solution is important for their analysis because it ensures the smoothness of the solution where it is positive and it implies its boundedness. Bernis, Peletier and Williams ([6]) studied the question whether self-similar source-type solutions of the Cauchy problem corresponding to (P_T) exist. They showed that only for $n \in (0,3)$ self-similar source-type solutions with finite mass exist.

A new idea in the study of fourth-order degenerate parabolic equations was the discovery of new integral (or entropy) estimates, which in one space dimension are derived from the equality

$$\tfrac{1}{\alpha(\alpha+1)} \frac{d}{dt} \int_\Omega u^{\alpha+1} = -\int_\Omega u^{n+\alpha-1} u_{xx}^2 - \tfrac{(n+\alpha-1)(2-\alpha-n)}{3} \int_\Omega u^{n+\alpha-3} u_x^4. \quad \text{(E)}$$

A careful analysis shows that identity (E) gives a priori estimates for real numbers α satisfying $\frac{1}{2} \leq \alpha + n \leq 2$. In the paper by Bernis and Friedman ([5]) the identity (E) was applied only for $\alpha = 1-n$. Using the new estimates Beretta, Bertsch and Dal Passo ([1]) and Bertozzi and Pugh ([7]) were able to prove regularity results that are optimal in the sense that they are sharp for the source-type similarity solutions. Integral estimates derived from a local version of identity (E) are used by Bernis ([3]) and Kersner and Shiskov ([16]) to show that in one space dimension solutions to (P_T) have the property of finite speed of propagation of their support if $n \in (0,2)$. In addition, Bernis ([3]) obtained regularity results for the resulting free boundary and decay estimates for $t \to \infty$. In a recent paper, Bernis ([4]) extends his result on finite speed of propagation to the case $n \in [2,3)$.

Although there has been some progress in the study of degenerate parabolic equations of type (P_T) in one space dimension, many questions are still unanswered. Probably the most important one is to define classes of functions in which problem (P_T) has a unique solution. In this context we refer to Beretta, Bertsch and Dal Passo [1] for an example of nonuniqueness. If $0 < n < 3$, a class in which problem (P_T) may be well-posed consists of functions with a prescribed contact angle at the edge of the support. The integral estimates resulting from identity (E) imply that the solutions have a zero contact angle for almost all t. The only result for a nonzero contact angle that is known to the authors is by Otto ([20]) who shows in space dimension $N = 1$ for the special case of growth exponent $n = 1$ existence of weak solutions with a prescribed contact angle.

In higher space dimensions there are existence results for degenerate parabolic equations of fourth order by Elliott and Garcke ([10, 11]) and Grün ([14]). But in these papers the growth exponent n was restricted to the interval $[1, 2)$ if one wants to prescribe initial data with compact support. In a recent paper Dal Passo, Garcke and Grün ([9]) were able to show a global version of the integral estimates in higher space dimensions (cf. also Section 2). These estimates make it possible to show existence of weak solutions to problem (P_T) if $n \in (\frac{1}{8}, 3)$. Furthermore, the integral estimates imply new regularity results; in particular, the zero contact angle is attained in a generalized sense (cf. Corollary 2.2 in [9]). But so far it is not known whether solutions to problem (P_T) in higher space dimensions are bounded or continuous. Recently Bernis and Ferreira have constructed self-similar source-type solutions in the case of higher space dimensions ([12]).

Let us briefly describe the outline of this paper. In Section 2, we present—basically following the spirit of [9]—the main ingredients and results concerning the construction of solutions to problem (P_T), using an approach that will be well adapted to the proof of a local version of the global integral estimates derived in [9]. These local integral estimates will be essential for our results about the qualitative behaviour of the solution's support (cf. Section 3).

For technical reasons we need to assume for this construction and therefore throughout the paper that $N = 2, 3$ and

$$n > \tfrac{1}{8}, \quad \text{if } N = 2, \quad n \in (\tfrac{1}{8}, 4), \quad \text{if } N = 3. \tag{H}$$

Of course our construction would work in one space dimension as well, but since this was already considered by other authors (cf. [1, 7, 3]), we do not state results in one space dimension. A priori estimates also hold for $n \in (0, \frac{1}{8})$, but they do not guarantee sufficient regularity to formulate a weak notion of solutions (cf. the proof of Theorem 2.1 in [9]). In space dimension three we can only prove integrability of the flux $u^n \nabla \Delta u$ if $0 < n < 4$ (cf. [9]), and therefore compactness of approximate solutions can only be established for this range of values of n.

In Section 3 we prove a local version of the integral estimates derived in [9] and we use these estimates to obtain a nonshrinking property for the solution's support under the assumption that $n > 3/2$. Another consequence of the local integral estimates is that the solution we constructed has finite speed of propagation. By this we mean that if a solution is zero in a ball $B_{r_0}(X_0) \subset \Omega$ at time t_0 it will remain zero in a slightly smaller ball for slightly later times $t > t_0$. This result is achieved by a generalization of

Bernis' technique ([3]) to higher space dimensions together with the methods derived by Dal Passo, Garcke and Grün ([9]) (cf. Section 4). Having shown finite speed of propagation it is possible to construct solutions to the Cauchy problem related to (P_T) that have compact support for all times $t \geq 0$ (Section 5). In addition, we prove decay estimates for these solutions and give upper estimates for the growth of their support. The rest of the paper is devoted to the proof of auxiliary results of interpolation and differential inequality type (see Section 6 and Appendix).

Notation and general assumptions. We assume that $\Omega \subset \mathbb{R}^N (N = 2, 3)$ is an open and bounded domain with boundary of class $C^{1,1}$ (or $C^{0,1}$ if Ω is convex) that is piecewise smooth. Therefore, the unit outward normal $\nu(x)$ to Ω exists for almost all $x \in \partial\Omega$. The initial data $u_0 \in H^1(\Omega)$ (or $u_0 \in H^1(\mathbb{R}^N)$ if we consider the Cauchy problem) are assumed to be nonnegative. By Ω_T we denote the space–time cylinder $\Omega \times (0,T)$. $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{(H^1)' \times H^1}$ stands for the duality product of a functional in $\big(H^1(\Omega)\big)'$ and an element in $H^1(\Omega)$. For a $(N \times N)$-matrix A and vectors $a, b \in \mathbb{R}^N$ we define $\langle a, A, b \rangle := \sum_{i,j=1}^N a_i A_{ij} b_j$. We define $[\zeta > 0] := \{x \in \Omega : \zeta(x) > 0\}$ and $B_{r_0}(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < r_0\}$. In addition we use the constants convention; i.e., different constants appearing in a sequence of inequalities may be denoted by the same symbol.

2. Construction of a solution. In this section, we briefly describe the main ingredients of the approximation process used to construct solutions to problem (P_T). This approximation method will be consistent with our technique to derive a local version of the integral estimates of Dal Passo, Garcke and Grün ([9]). Let us consider for $\sigma > 0$, $\delta > 0$ auxiliary problems

$$\begin{cases} (u_{\sigma\delta})_t + \operatorname{div}\big(m_{\sigma\delta}(u_{\sigma\delta})\nabla\Delta u_{\sigma\delta}\big) = 0 & \text{in } \Omega \times (0,T) \\ \frac{\partial}{\partial\nu} u_{\sigma\delta} = \frac{\partial}{\partial\nu}\Delta u_{\sigma\delta} = 0 & \text{on } \partial\Omega \times [0,T] \\ u_{\sigma\delta}(0) = u_{0\sigma\delta} := u_0 + \delta^{\theta_1} + \sigma^{\theta_2} & \text{in } \Omega, \end{cases} \qquad (P_{T,\sigma\delta})$$

where $u_0 \in H^1(\Omega)$ is nonnegative and θ_1, θ_2 are positive real numbers. For the diffusion coefficient $m_{\sigma\delta}$ we choose

$$m_{\sigma\delta}(\tau) := \frac{\tau^{n+s}}{\delta\tau^n + \tau^s + \sigma\tau^{n+s}},$$

and we assume that $\int_\Omega u_0^{\alpha+1} < \infty$ for a number $\alpha \in \big(\frac{1}{2} - n, 2 - n\big)$.

If s is sufficiently large, our particular choice of bounded diffusion coefficients $m_{\sigma\delta}$ ensures that there exists a solution to problem $(P_{T,\sigma\delta})$ which is positive for almost every $t \in (0,T)$. In addition, the integral estimates of

Proposition 1.2 of [9] hold true, and for almost every $t \in (0,T)$ we have that $\nabla\Delta u(t,.)$ is contained in $L^2(\Omega)$.

Passing successively first with δ and then with σ to zero, we obtain, following the lines of proof of Theorem 2.1 of [9], a solution of (P_T) in the sense of

Definition 2.1. A solution of problem (P_T) is a nonnegative function $u \in L^\infty(0,T;H^1(\Omega))$ with the following properties:

i) for all $q > \frac{4N}{2N+(2-N)n}$, $u \in H^{1,2}(0,T;(H^{1,q}(\Omega))')$,

ii) there exists an $\alpha \in (\frac{1}{2}-n, 2-n)$ such that

$$u^{\frac{\alpha+n+1}{4}} \in L^4(0,T;H^{1,4}(\Omega)), \quad \text{and} \quad u^{\frac{\alpha+n+1}{2}} \in L^2(0,T;H^{2,2}(\Omega)),$$

iii) u solves the equation $u_t + \operatorname{div}(u^n\nabla\Delta u) = 0$ in the sense that

$$\int_0^T \langle u_t(t),\zeta(t)\rangle = \tfrac{1}{2}\int_{[u>0]} n(n-1)u^{n-2}\,|\nabla u|^2\,\nabla u\nabla\zeta$$
$$+\tfrac{1}{2}\int_{[u>0]} nu^{n-1}\,|\nabla u|^2\,\Delta\zeta + \int_{[u>0]} nu^{n-1}\big\langle\nabla u, D^2\zeta,\nabla u\big\rangle + \int_{\Omega_T} u^n\nabla u\nabla\Delta\zeta$$

for all $\zeta \in L^\infty\big(0,T;H^{3,\infty}(\Omega)\big)$ fulfilling $\nabla\zeta\cdot\nu = 0$ on $\partial\Omega\times[0,T]$.

iv) u attains its initial data in the sense $\lim_{t\searrow 0} u(t) = u_0$ in $L^1(\Omega)$.

Let us collect some properties of the solution u. The regularity of u implies that $u \in C([0,T];L^p(\Omega))$ for $p \in [1,\infty)$ if $N=2$ and for $p\in[1,6)$ if $N=3$ (cf. Corollary 4 in Simon [21]). Hence, u attains its initial values in the sense of iv). Since $u \in C([0,T];(H^{1,q}(\Omega))')\cap L^\infty(0,T;H^1(\Omega))$ for $q > \frac{4N}{2N+(2-N)n}$ we can conclude that $u(t)$ is well-defined in $H^1(\Omega)$ for all $t\in[0,T]$. In addition, we obtain $u \in C_S([0,T];H^1(\Omega))$, where $C_S([0,T];H^1(\Omega))$ denotes the space of all functions $u:[0,T]\to H^1(\Omega)$ that are continuous in t with respect to the weak topology in $H^1(\Omega)$. For the two latter results we refer to Lions and Magenes [17, Chapter 3, Lemma 8.1] (see also [3]). Another consequence of $u\in C_S([0,T];H^1(\Omega))$ is that $t\mapsto\int_\Omega|\nabla u|^2(t)$ is lower semi-continuous. Following the arguments of Dal Passo, Garcke and Grün ([9; cf. the proof of Theorem 2.4]) we can establish $\int_\Omega|\nabla u|^2(t_2)\le\int_\Omega|\nabla u|^2(t_1)$ for almost all $t_1,t_2\in[0,T]$ with $t_1<t_2$.

Now we state a global integral estimate that is valid for all real numbers α satisfying $\frac{1}{2}<\alpha+n<2$, $\alpha\neq 0,-1$ and all $t\in(0,T)$:

$$\begin{aligned}\tfrac{1}{\alpha(\alpha+1)}\int_\Omega u^{\alpha+1}(t) + C_1^{-1}\Big\{\int_{\Omega_t}\big|D^2u^{\frac{\alpha+n+1}{2}}\big|^2 + \int_{\Omega_t}\big|\nabla u^{\frac{\alpha+n+1}{4}}\big|^4\Big\}&\\ \le \tfrac{1}{\alpha(\alpha+1)}\int_\Omega u_0^{\alpha+1} + C_2\int_{\Omega_t}u^{\alpha+n+1},&\end{aligned}\tag{1}$$

where C_1 depends on α and n and C_2 depends on α, n, N and Ω. For the proof we refer to Dal Passo, Garcke and Grün [9]. We want to point out that $\int_\Omega u_0^{\alpha+1}$ is bounded for compactly supported initial data only if $\alpha + 1 > 0$. Hence, we obtain that for arbitrary nonnegative initial data and $n \in (0, 3)$ the regularity stated in ii) holds for all $\alpha \in \big(\max\{\frac{1}{2} - n, -1\}, 2 - n\big)$. It was shown in [9] that $C_2 = 0$ if the domain Ω is convex.

Applying a diagonal procedure or continuing successively the solution we may also replace the interval $(0, T)$ by $(0, \infty)$ to get a solution on $\Omega \times (0, \infty)$. We refer to the initial value problem on $\Omega \times (0, \infty)$ by problem (P) or by problem $(P_{\sigma\delta})$ in the case of the modified problem.

3. Integral estimates—local version. In this section we derive a local version of the integral estimate (1) and show local positivity properties of the solution. In addition, we prove a nonshrinking property for the support of solutions if n is large enough.

Let us now formulate the local version of the integral estimate.

Theorem 3.1. *Let hypothesis* (H) *be fulfilled, let u be a solution of problem (P) constructed as in Section* 2 *and let α be a real number satisfying $\frac{1}{2} < \alpha + n < 2$ and $\alpha \neq 0, -1$. Let $\zeta \in C_0^2(\Omega)$ be a nonnegative function such that $\int_\Omega \zeta^4 u_0^{\alpha+1} < \infty$.*

Then there exists a positive constant C_1 depending only on α and n, such that the following estimate is valid for all $t \in (0, \infty)$:

$$\tfrac{1}{\alpha(\alpha+1)} \int_\Omega \zeta^4 \, u^{\alpha+1}(t) + C_1^{-1} \Big\{ \int_{\Omega_t} \zeta^4 \, \big| D^2 u^{\frac{\alpha+n+1}{2}} \big|^2 + \int_{\Omega_t} \zeta^4 \, \big| \nabla u^{\frac{\alpha+n+1}{4}} \big|^4 \Big\}$$
$$\leq \tfrac{1}{\alpha(\alpha+1)} \int_\Omega \zeta^4 u_0^{\alpha+1} + C_1 \int_0^t \int_{[\zeta>0]} u^{\alpha+n+1} \big(|\nabla\zeta|^4 + \zeta^2 |\Delta\zeta|^2 \big). \qquad (2)$$

Proof. The proof consists of two steps. First, we prove an estimate similar to (2) for a solution $u_{\sigma\delta}$ of Problem $(P_{\sigma\delta})$. Afterwards we pass to the limits $\delta \searrow 0$ and $\sigma \searrow 0$.

For each σ, δ we consider the functions

$$g_{\sigma\delta}^\alpha(\tau) = \tfrac{\delta}{\alpha+n-s} \tau^{\alpha+n-s} + \tfrac{1}{\alpha} \tau^\alpha + \tfrac{\sigma}{\alpha+n} \tau^{\alpha+n} \qquad (3)$$

and

$$G_{\sigma\delta}^\alpha(\tau) = \tfrac{\delta}{(\alpha+n-s)(\alpha+n-s+1)} \tau^{\alpha+n-s+1} + \tfrac{1}{\alpha(\alpha+1)} \tau^{\alpha+1} + \tfrac{\sigma}{(\alpha+n)(\alpha+n+1)} \tau^{\alpha+n+1}, \qquad (4)$$

which have the property $(G_{\sigma\delta}^\alpha)'' = (g_{\sigma\delta}^\alpha)' = \frac{\tau^{\alpha+n-1}}{m_{\sigma\delta}(\tau)}$. We choose s large enough such that we can conclude from Lemma 2.1 of [9] that $u(t)$ is strictly positive for almost all $t \in (0,\infty)$. Furthermore, $\nabla\Delta u(t)$ exists for these t as an element of $L^2(\Omega)$. Hence, $u_{\sigma\delta}$ fulfills for all $T > 0$ and all $\phi \in L^2\big(0,T;H^1(\Omega)\big)$ the identity

$$\int_0^T \langle (u_{\sigma\delta})_t, \phi\rangle - \int_{\Omega_T} m_{\sigma\delta}(u_{\sigma\delta})\nabla\Delta u_{\sigma\delta}\cdot\nabla\phi = 0. \tag{5}$$

Let us now choose $\phi = \zeta^4\, g^\alpha_{\sigma\delta}(u_{\sigma\delta}+\varepsilon)$ as test function in (5). For the parabolic part (i.e., for the first term in (5)) we obtain

$$\int_0^T \zeta^4\big\langle \big(u_{\sigma\delta}(t)\big)_t, g^\alpha_{\sigma\delta}\big(u_{\sigma\delta}(t)+\varepsilon\big)\big\rangle = \int_\Omega \zeta^4\, G^\alpha_{\sigma\delta}(u_{\sigma\delta}+\varepsilon)(T) - \int_\Omega \zeta^4\, G^\alpha_{\sigma\delta}(u_{0\sigma\delta}+\varepsilon). \tag{6}$$

For the elliptic part we estimate in a way similar to the proof of Proposition 1.2 of [9] for almost all $t \in (0,T)$:

$$\begin{aligned}
&-\int_\Omega m_{\sigma\delta}(u_{\sigma\delta})\nabla\Delta u_{\sigma\delta}\,\nabla\left(\zeta^4 g^\alpha_{\sigma\delta}(u_{\sigma\delta}+\varepsilon)\right)\\
&\quad\geq \gamma^{-2}\int_\Omega \zeta^4(u_{\sigma\delta}+\varepsilon)^{\alpha+n+1-2\gamma}\Big\{\tfrac{2}{3}\big[D^2(u_{\sigma\delta}+\varepsilon)^\gamma\big]^2 + \tfrac{1}{3}\big[\Delta(u_{\sigma\delta}+\varepsilon)^\gamma\big]^2\Big\}\\
&\qquad + c(\alpha,n,\gamma)\int_\Omega \zeta^4(u_{\sigma\delta}+\varepsilon)^{\alpha+n-3}|\nabla u_{\sigma\delta}|^4\\
&\qquad - (\delta_1+\delta_2)\int_\Omega \zeta^4(u_{\sigma\delta}+\varepsilon)^{\alpha+n-3}|\nabla u_{\sigma\delta}|^4\\
&\qquad - \tfrac{C}{\delta_1}\int_\Omega \zeta^4(u_{\sigma\delta}+\varepsilon)^{\alpha+n-1}\Big(1-\tfrac{m_{\sigma\delta}(u_{\sigma\delta})}{m_{\sigma\delta}(u_{\sigma\delta}+\varepsilon)}\Big)|\Delta u_{\sigma\delta}|^2\\
&\qquad - \tfrac{C}{\delta_2}\int_\Omega \zeta^4(u_{\sigma\delta}+\varepsilon)^{\alpha+n-1}|\Delta u_{\sigma\delta}|^2\Big(\frac{\varepsilon}{u_{\sigma\delta}+\varepsilon}\Big)^2\\
&\qquad - \int_\Omega m_{\sigma\delta}(u_{\sigma\delta})\, g^\alpha_{\sigma\delta}(u_{\sigma\delta}+\varepsilon)\,\nabla\Delta u_{\sigma\delta}\nabla\zeta^4\\
&\qquad - \int_\Omega \Big(\tfrac{m_{\sigma\delta}(u_{\sigma\delta})}{m_{\sigma\delta}(u_{\sigma\delta}+\varepsilon)}+1\Big)(u_{\sigma\delta}+\varepsilon)^{\alpha+n-1}|\Delta u_{\sigma\delta}|\,|\nabla u_{\sigma\delta}|\,|\nabla\zeta^4|\\
&\qquad - \int_\Omega (u_{\sigma\delta}+\varepsilon)^{\alpha+n-1}|\nabla u_{\sigma\delta}||D^2 u_{\sigma\delta}||\nabla\zeta^4|\\
&\qquad - \int_\Omega (u_{\sigma\delta}+\varepsilon)^{\alpha+n-2}|\nabla u_{\sigma\delta}|^3|\nabla\zeta^4|\\
&\quad =: I_1 + I_2 + A_1 + A_2 + A_3 + L_1 + L_2 + L_3 + L_4
\end{aligned} \tag{7}$$

with the notation

$$c(\alpha,n,\gamma) := (1-\gamma)(\gamma-\tfrac{1}{3}) - \tfrac{1}{3}(\alpha+n-1)(\alpha+n-2) + \tfrac{2}{3}(1-\gamma)(1-\alpha-n), \tag{8}$$

where γ can be chosen such that $c(\alpha, n, \gamma)$ is positive (cf. [1], [9]). Hence, the terms I_1 and I_2 are positive and it remains to estimate A_1, A_2, A_3 and L_1, L_2, L_3, L_4. The term A_1 can be absorbed in I_2 if we choose δ_1 and δ_2 small enough. Using the global integral estimates derived for $u_{\sigma\delta}$ in Section 2 we obtain that, for σ, δ fixed and when integrated over time, $A_2 + A_3 = o_\varepsilon(1)$ as ε tends to zero.

Now we estimate the terms L_1 to L_4. First we get

$$\begin{aligned} L_1 = & \int_\Omega m'_{\sigma\delta}(u_{\sigma\delta})\, g^\alpha_{\sigma\delta}(u_{\sigma\delta} + \varepsilon)\, \Delta u_{\sigma\delta}\, \nabla u_{\sigma\delta}\, \nabla \zeta^4 \\ & + \int_\Omega m_{\sigma\delta}(u_{\sigma\delta})(g^\alpha_{\sigma\delta})'(u_{\sigma\delta} + \varepsilon)\, \Delta u_{\sigma\delta}\, \nabla u_{\sigma\delta}\, \nabla \zeta^4 \\ & + \int_\Omega m_{\sigma\delta}(u_{\sigma\delta}) g^\alpha_{\sigma\delta}(u_{\sigma\delta} + \varepsilon)\, \Delta u_{\sigma\delta}\, \Delta \zeta^4 =: L_1^1 + L_1^2 + L_1^3. \end{aligned} \tag{9}$$

In the Appendix we shall prove that there exists a constant C (independent of σ, δ) and a constant $\hat{C}(\sigma, \delta)$ such that the following estimates hold for all $\tau \geq 0$ and all $\varepsilon \geq 0$:

$$\begin{aligned} m'_{\sigma\delta}(\tau)\, g^\alpha_{\sigma\delta}(\tau + \varepsilon) &\leq C|\tau + \varepsilon|^{\alpha+n-1} + \hat{C}(\sigma, \delta)\, \varepsilon |\tau + \varepsilon|^{\alpha+n-1}, \\ m_{\sigma\delta}(\tau)\, (g^\alpha_{\sigma\delta})'(\tau + \varepsilon) &\leq C|\tau + \varepsilon|^{\alpha+n-1}, \\ m_{\sigma\delta}(\tau)\, g^\alpha_{\sigma\delta}(\tau + \varepsilon) &\leq C|\tau + \varepsilon|^{\alpha+n}. \end{aligned} \tag{10}$$

Thus we obtain

$$\begin{aligned} |L_1^1| + |L_1^2| + |L_2| + |L_3| &\leq C \int_\Omega (u_{\sigma\delta} + \varepsilon)^{\alpha+n-1} |\nabla u_{\sigma\delta}||D^2 u_{\sigma\delta}|\zeta^3 |\nabla \zeta| + o_\varepsilon(1) \\ \leq \delta_3 \Big\{ & \int_\Omega \zeta^4 \big|\nabla (u_{\sigma\delta} + \varepsilon)^{\frac{\alpha+n+1}{4}}\big|^4 + \int_\Omega \zeta^4 \big|D^2 (u_{\sigma\delta} + \varepsilon)^{\frac{\alpha+n+1}{2}}\big|^2 \Big\} \\ & + C_{\delta_3} \int_{[\zeta > 0]} (u_{\sigma\delta} + \varepsilon)^{\alpha+n+1}\, |\nabla \zeta|^4 + o_\varepsilon(1). \end{aligned}$$

For L_1^3 we get

$$\begin{aligned} |L_1^3| &\leq C \int_\Omega (u_{\sigma\delta} + \varepsilon)^{\alpha+n} |\Delta u_{\sigma\delta}|\, \zeta^2 |\nabla \zeta|^2 + C \int_\Omega (u_{\sigma\delta} + \varepsilon)^{\alpha+n}\, |\Delta u_{\sigma\delta}|\, \zeta^3 |\Delta \zeta| \\ &\leq \delta_4 \int_\Omega (u_{\sigma\delta} + \varepsilon)^{\alpha+n-1} |\Delta u_{\sigma\delta}|^2 \zeta^4 + C_{\delta_4} \int_{[\zeta>0]} (u_{\sigma\delta} + \varepsilon)^{\alpha+n+1} \big(|\nabla \zeta|^4 + \zeta^2 |\Delta \zeta|^2\big). \end{aligned}$$

Finally, we estimate the last term:

$$\begin{aligned} |L_4| &\leq C \int_\Omega (u_{\sigma\delta} + \varepsilon)^{\alpha+n-2}\, |\nabla u_{\sigma\delta}|^3\, \zeta^3 |\nabla \zeta| \\ &\leq \delta_5 \int_\Omega \zeta^4 (u_{\sigma\delta} + \varepsilon)^{\alpha+n-3} |\nabla u_{\sigma\delta}|^4 + C_{\delta_5} \int_{[\zeta>0]} (u_{\sigma\delta} + \varepsilon)^{\alpha+n+1} |\nabla \zeta|^4. \end{aligned}$$

Now one can use the inequalities

$$\begin{aligned}&\int_\Omega \zeta^4 (u_{\sigma\delta}+\varepsilon)^{\alpha+n-1} |D^2 u_{\sigma\delta}|^2 \\ \le &\hat{C}\Big\{\int_\Omega \zeta^4 \big|D^2 (u_{\sigma\delta}+\varepsilon)^{\frac{\alpha+n+1}{2}}\big|^2 + \int_\Omega \zeta^4 \big|\nabla (u_{\sigma\delta}+\varepsilon)^{\frac{\alpha+n+1}{4}}\big|^4\Big\}\end{aligned} \tag{11}$$

and

$$\begin{aligned}&\int_\Omega \zeta^4 \big|D^2 (u_{\sigma\delta}+\varepsilon)^{\alpha+n+1}\big|^2 \\ \le &\hat{C}\Big\{\int_\Omega \zeta^4 u_{\sigma\delta}^{\alpha+n+1-2\gamma} \big|D^2 (u_{\sigma\delta}+\varepsilon)^{\gamma}\big|^2 + \int_\Omega \zeta^4 \big|\nabla (u_{\sigma\delta}+\varepsilon)^{\frac{\alpha+n+1}{4}}\big|^4\Big\}\end{aligned} \tag{12}$$

to control the terms $L_1, \dots, L_4$. The estimate for the elliptic part together with (5), (6), (11) and (12) gives (2) if we pass to the limits $\varepsilon \searrow 0, \delta \searrow 0$ and $\sigma \searrow 0$.

Remark 3.2. If we assume in addition to the assumptions in Theorem 3.1 that α is positive, we can derive for all $0 \le t_1 < t_2 < \infty$

$$\begin{aligned}\int_\Omega \zeta^4\, u^{\alpha+1}(t_2) + C_2^{-1}\Big\{\int_{t_1}^{t_2}\int_\Omega \zeta^4 \big|D^2 u^{\frac{\alpha+n+1}{2}}\big|^2 + \int_{t_1}^{t_2}\int_\Omega \zeta^4 \big|\nabla u^{\frac{\alpha+n+1}{4}}\big|^4\Big\} \\ \le \int_\Omega \zeta^4 u^{\alpha+1}(t_1) + C_2 \int_{t_1}^{t_2}\int_{[\zeta>0]} u^{\alpha+n+1}\big(|\nabla\zeta|^4 + \zeta^2 |\Delta\zeta|^2\big),\end{aligned} \tag{13}$$

with a constant C_2 depending on α and n. This estimate holds for all $\zeta \in C_0^2(\Omega)$ and for $\zeta \equiv 1$ if Ω is convex.

To show (13) we choose $\phi(x,t) = \chi(t)\zeta^4(x)\, g_{\sigma\delta}^{\alpha}(u_{\sigma\delta}(x,t)+\varepsilon)$ in (5), where χ is the characteristic function of the interval $[t_1, t_2]$. Then we proceed as in the proof of Theorem 3.1. But in order to pass to the limit $\delta \searrow 0$ in the term $\int_\Omega G_{\sigma\delta}^{\alpha}(u_{\sigma\delta})(t_1)$ we need to show

$$\delta \int_\Omega u_{\sigma\delta}^{\alpha+n-s+1}(t_1) \to 0 \text{ as } \delta \searrow 0.$$

In the case $t_1 = 0$ we could use that $u_{0,\sigma}$ is uniformly bounded away from zero to proceed. Since the global integral estimate has been established for a whole interval of values of α, we can use a similar argument as was used in the proof of Lemma 3.3 ([3]) to show the above convergence in the case $t_1 > 0$ as well. To prove the convergence for a fixed α we use that we have a uniform bound on $\delta \int_\Omega u_{\sigma\delta}^{\tilde{\alpha}+n-s+1}(t_1)$ for a $\tilde{\alpha} < \alpha$ to control $\delta u_{\sigma\delta}^{\alpha+n-s+1}$ at points where u_σ is small. The convergence of all other terms can be shown as in the proof of Theorem 3.1.

The local integral estimate as formulated in Theorem 3.1 enables us to generalize results of Beretta, Bertsch and Dal Passo ([1]) on positivity properties to space dimensions two and three. Our results improve results by Grün ([14]) who first showed positivity properties in higher space dimensions.

Corollary 3.3. *Let hypothesis* (H) *be fulfilled and let* u *be a solution of Problem* (P) *constructed as in Section* 2. *Assume* $\zeta \in C_0^2(\Omega)$ *fulfills* $\int_\Omega \zeta^4 u_0^{\frac{3}{2}-n} < \infty$.

Then u *has the following properties:*

i) *if* $n > \frac{3}{2}$ *then for all Lebesgue measurable sets* $E \subset [\zeta > 0]$ *with positive measure*

$$\int_E u(x,t)\,dx > 0, \ \textit{for all } t \in (0,\infty),$$

ii) *if* $N = 2$ *and* $n > 3$ *then for almost all* $t \in (0,\infty)$ $u(t)$ *is strictly positive on* $[\zeta > 0]$.

Proof. Assume $\int_E u(x,t)\,dx = 0$ for a set $E \subset [\zeta > 0]$ with positive measure and a time $t > 0$. Then $u(t)$ would be zero on a set with positive measure. But with the help of Theorem 3.1 we control $\int_\Omega \zeta^4 u^{\alpha+1}(t)$ for all α with $\frac{3}{2} - n < \alpha + 1 < 3 - n$. Since $\alpha + 1$ can be chosen negative, this gives a contradiction.

It remains to prove ii). We choose an arbitrary open ball $B \subset\subset [\zeta > 0]$. Theorem 3.1 and the $L^\infty(0,\infty; H^1(\Omega))$-regularity of u implies that $u^{\frac{\alpha+n+1}{4}}(t) \in H^{1,4}(B)$ for all α with $\frac{1}{2} < \alpha + n < 2$ and almost all $t \in (0,\infty)$. Hence, for these t it holds that $u^{\frac{\alpha+n+1}{4}}(t) \in C^\gamma(B)$ if $\gamma \in (0, \frac{4-N}{4})$. Assume there exists a $x_0 \in B$ with $u(x_0,t) = 0$. Using the inequality $u(x,t)^{\frac{\alpha+n+1}{4}} < C|x - x_0|^\gamma \quad (x \in B)$ and estimate (2) of Theorem 3.1 we get for $\alpha + 1 < 0$

$$\infty > \int_\Omega \zeta^4\, u(x,t)^{\alpha+1}\,dx > C^{-1} \int_B \zeta^4 |x - x_0|^{\frac{4\gamma(\alpha+1)}{\alpha+n+1}}\,dx.$$

This gives a contradiction if $n > \frac{6}{4-N}$, because we can choose α such that the integral on the right-hand side becomes unbounded. Therefore, $u(t)$ is strictly positive on B for almost all t. Applying the above argument for a countable collection of balls $B \subset\subset [\zeta > 0]$ that cover $[\zeta > 0]$ gives the result.

4. Finite speed of propagation. In this section we show that the solution we constructed in Section 2 has the property of finite speed of propagation of its support if $n \in (\frac{1}{8}, 2)$ and the space dimension is two or three. First we give a definition of finite speed of propagation.

Definition 4.1. We say a function $v : \Omega \times (0, \infty) \to \mathbb{R}$ has *finite speed of propagation* if for all $t_0 > 0$, $x_0 \in \Omega$ and $r_0 > 0$ with $B_{r_0}(x_0) \subset \Omega$ and $v(t_0) \equiv 0$ almost everywhere in $B_{r_0}(x_0)$ there exists a $T_* > 0$ and a continuous function $r : [t_0, t_0 + T_*) \to \mathbb{R}^+$ with $r(t_0) = r_0$ such that $v(x,t) = 0$ for all $t \in [t_0, t_0 + T_*)$ and $x \in B_{r(t)}(x_0)$.

The following result states that the solution constructed in Section 2 has finite speed of propagation if $n \in (\frac{1}{8}, 2)$.

Theorem 4.2. *Let $n \in (\frac{1}{8}, 2)$, $N \in \{2, 3\}$ and $\alpha \in (\max\{\frac{1}{2} - n, 0\}, 2 - n)$. Let u be the solution of problem (P) constructed in Section 2 and assume that there exists a time $t_0 \geq 0$ such that $u(t_0) \equiv 0$ in $B_{r_0}(x_0) \subset \Omega$ almost everywhere. Then there exists a positive constant T_* depending on α, n, N, r_0 and $u(t_0)$ such that for all $t \in (t_0, t_0 + T_*)$, $u(t) \equiv 0$ a.e. in $B_{r(t)}(x_0)$, where $r(t)$ is defined through*

$$(r(t))^N = (r_0)^N - A_0 (t - t_0)^{\eta/4} \left(\int_{t_0}^{t} \int_{B_{r_0}(x_0)} \left| D^2 u^{\frac{\alpha+n+1}{2}} \right|^2 \right)^{\beta}. \qquad (14)$$

The constant A_0 depends on α, n, r_0 and N and the exponents η and β are

$$\eta = \frac{4(\alpha+1)}{4N\alpha + 5Nn + 4N - 4n} \quad \text{and} \quad \beta = \frac{n}{4N\alpha + 5Nn + 4N - 4n}.$$

Proof. The proof follows the line of Bernis' proof in one space dimension ([3]). Since the technical modifications are not straightforward, we present the proof. For simplicity we assume without loss of generality that $(x_0, t_0) = (0, 0)$ (see Remark 3.2). In order to proceed we need to derive the estimate (2) of Theorem 3.1 for $\zeta(x) = \phi_r(x) = (r^N - |x|^N)_+$ with $r > 0$. We can derive this estimate if we approximate ϕ_r by smooth functions and pass to the limit in the estimate for the smoother functions.

For ϕ_r the following estimates hold:

$$|\nabla \phi_r(x)| \leq N\, r^{N-1}, \text{ if } |x| < r$$

and

$$|\Delta \phi_r(x)| \leq 2N(N-1)\, r^{N-2}, \text{ if } |x| < r.$$

Using the notation $w = u^{\frac{\alpha+n+1}{2}}$ and $q = 2 - \frac{2n}{\alpha+n+1}$ (cf. Bernis [3]) we can derive from Theorem 3.1 that there exists a constant C independent of r

such that for all $r \in (0, r_0)$

$$\sup_{0<t<T} \int_{B_r(0)} \left(r^N - |x|^N\right)^4 w^q + C^{-1} \int_0^T \int_{B_r(0)} \left(r^N - |x|^N\right)^4 |D^2 w|^2 \le C \int_0^T \int_{B_r(0)} r^{4(N-1)} w^2. \tag{15}$$

Now we define

$$E_s(r,T) := \int_0^T \int_{B_r(0)} \left(r^N - |x|^N\right)^s |D^2 w|^2,$$
$$F(r,T) := \sup_{0<t<T} \int_{B_r(0)} \left(r^N - |x|^N\right)^4 w^q$$

and obtain, using (15) and a weighted version of the Gagliardo–Nirenberg inequality (cf. Lemma 6.3),

$$\begin{aligned} F + E_4 &\le C r^{4(N-1)} \int_0^T \int_{B_r(0)} w^2 \\ &\le C r^{4(N-1)} \int_0^T \left(\int_{B_r(0)} |D^2 w|^2\right)^d \left(\int_{B_r(0)} \left(r^N - |x|^N\right)^4 w^q\right)^{\frac{2}{q}(1-d)} \\ &\quad + C r^{-2\nu + 4(N-1)} \int_0^T \left(\int_{B_r(0)} \left(r^N - |x|^N\right)^4 w^q\right)^{\frac{2}{q}} \\ &\le C r^{4(N-1)} E_0^d T^{1-d} F^{\frac{2}{q}(1-d)} + C r^{-2\nu+4(N-1)} T F^{\frac{2}{q}}. \end{aligned}$$

Hence, we may choose T_0 so small that $C r^{-2\nu+4(N-1)} T F^{\frac{2}{q}}(r,T) \le F(r,T)/2$ for $r \in (\frac{r_0}{2}, r_0)$ and $T \in (0, T_0)$. This implies that for these (r,T) we have

$$F(r,T) + E_4(r,T) \le C r^{4(N-1)} E_0^d(r,T) T^{1-d} F^{\frac{2}{q}(1-d)}(r,T).$$

Now Young's inequality yields

$$F(r,T) + E_4(r,T) \le C r^\kappa E_0^\theta(r,T) T^\eta \tag{16}$$

with

$$\kappa = \frac{4(N-1)q}{q - 2(1-d)}, \quad \theta = \frac{dq}{q-2(1-d)}, \quad \eta = \frac{(1-d)q}{q-2(1-d)}.$$

For $r \in (\frac{r_0}{2}, r_0)$ this leads to the following differential-type inequality for E_0:

$$E_4(r,T) \le C \cdot (r_0)^\kappa E_0^\theta T^\eta.$$

Since $\theta > 1$ it follows from Lemma 6.4 with $K = C \cdot (r_0)^\kappa T^\eta$ that $E_0(r,T) = 0$ if $r \le r_1$, where r_1 is defined through

$$r_1^N = r_0^N - \tfrac{\theta+3}{\theta-1} C^{\frac{1}{4}} (r_0)^{\frac{\kappa}{4}} T^{\frac{\eta}{4}} E_0(r_0,T)^{\frac{\theta-1}{4}}.$$

But this property only holds as long as $r_1 \ge \frac{r_0}{2}$.

With the help of the integral estimate (1) and the Gagliardo–Nirenberg inequality (cf. [19]) we estimate

$$\begin{aligned}
&E_0(r_0,T) \le \int_{\Omega_T} |D^2 u^{\frac{\alpha+n+1}{2}}|^2 \\
&\le \tfrac{C_1}{\alpha(\alpha+1)} \int_\Omega u_0^{\alpha+1} + C_1 C_2 \int_0^T \int_\Omega u^{\alpha+n+1} \\
&\le \tfrac{C_1}{\alpha(\alpha+1)} \int_\Omega u_0^{\alpha+1} + C \int_0^T \big(|\nabla u(t)|_2^{a(\alpha+n+1)} |u_0|_1^{(1-a)(\alpha+n+1)} + |u_0|_1^{\alpha+n+1} \big) \\
&\le \tfrac{C_1}{\alpha(\alpha+1)} \int_\Omega u_0^{\alpha+1} + CT \big(|\nabla u_0|_2^{a(\alpha+n+1)} |u_0|_1^{(1-a)(\alpha+n+1)} + |u_0|_1^{\alpha+n+1} \big),
\end{aligned}$$

where $a = (\alpha+n)\frac{2N}{N+2}$.

This implies that we can choose T so small that $r_1 \ge \frac{r_0}{2}$. For these T we conclude from (16) that $F(r_1,T) = 0$, and hence $u(T) \equiv 0$ almost everywhere on $B_{r_1}(0)$. Therefore, the proof of Theorem 4.2 is complete, and in particular we showed that u has finite speed of propagation.

Remark 4.3. If Ω is convex the integral estimate (13) with $\zeta \equiv 1$ can be applied to show that $\int_{t_0}^\infty \int_{B_{r_0}(x_0)} \big| D^2 u^{\frac{\alpha+n+1}{2}} \big|^2$ can be estimated by $\int_\Omega u^{\alpha+1}(t_0)$. This implies that for Ω convex the time T_*, which appeared in Theorem 4.2, only depends on α, n, N, r_0 and $\int_\Omega u^{\alpha+1}(t_0)$.

5. The Cauchy problem. The result on finite speed of propagation as formulated in Theorem 4.2 allows us to construct solutions to the Cauchy problem

$$\begin{cases} u_t + \operatorname{div}\big(u^n \nabla \Delta u\big) = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(0) = u_0 & \text{in } \mathbb{R}^N \end{cases} \tag{CP}$$

which are compactly supported for all $t \ge 0$.

Theorem 5.1. *Assume $n \in (\frac{1}{8}, 2)$, $N \in \{2,3\}$, and let $u_0 \in H^1(\mathbb{R}^N)$ be a nonnegative function with the property $u_0(x) = 0$ if $|x| > R_0$, where $R_0 \in \mathbb{R}^+$. Then there exists a nonnegative compactly supported solution $u \in L^\infty((0,\infty); H^1(\mathbb{R}^N))$ of* (CP) *in the sense that for all $T > 0$ there exists a $R(T) > 0$ such that*

i) *$u(x,t) = 0$ if $|x| > R(T)$ and $t \in [0,T]$,*

ii) *u is a solution of (P_T) with $\Omega = B_{2R(T)}(0)$ and initial value u_0.*
In addition, the following properties hold true:

iii) *there exists a constant c_1 depending on α and n such that for all $0 \le t_1 < t_2 < \infty$ and $\alpha \in (\max\{0, \frac{1}{2} - n\}, 2 - n)$*

$$\int_{\mathbb{R}^N} u^{\alpha+1}(t_2) + c_1\Big\{ \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |D^2 u^{\frac{\alpha+n+1}{2}}|^2 + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |\nabla u^{\frac{\alpha+n+1}{4}}|^4 \Big\} \le \int_{\mathbb{R}^N} u^{\alpha+1}(t_1), \tag{17}$$

iv) *the function $t \mapsto \int_{\mathbb{R}^N} |\nabla u|^2(t)$ is almost everywhere equal to a non-increasing function.*

Proof. We choose a real number $\bar{R} > 0$, set $\Omega = B_{R_0+4\bar{R}}(0)$ and construct a solution to (P_T) as described in Section 2. For x with $|x| > R_0 + 4\bar{R}$ and $t \in [0, T]$ we set $u(x,t) = 0$.

Covering the set $\Omega \setminus B_{R_0}$ by balls of radius $2\bar{R}$ and applying to these latter balls Theorem 4.2 it follows that there exists a time T_* depending on $\alpha, n, N, \bar{R}$ and $\int_{\mathbb{R}^N} u_0^{\alpha+1}$ (cf. Remark 4.3) such that $u(x,t) = 0$ if $|x| > R_0 + \bar{R}$ and $t \in [0, T_*]$. Now we choose $u(.,T_*)$ as new initial data. Repeating the above construction with R_0 replaced by $R_0 + \bar{R}$ and the initial time 0 replaced by T_* we obtain a solution in $B_{R_0+5\bar{R}} \times [T_*, 2T_*]$ satisfying $u(x,t) = 0$ if $|x| > R_0 + 2\bar{R}$ and $t \in [T_*, 2T_*]$. For the last step we used that

$$\int_{\mathbb{R}^N} u^{\alpha+1}(T_*) \le \int_{\mathbb{R}^N} u_0^{\alpha+1}$$

to conclude that the support of $u(t)$ remains in the set $B_{R_0+2\bar{R}}$ on the whole interval $[T_*, 2T_*]$ (cf. inequality (1) in which $C_2 = 0$ since Ω is convex).

Using the regularity properties of solutions to problem (P_T) (cf. Definition 2.1) we may “glue together” the solutions obtained on $[0, T_*]$ and $[T_*, 2T_*]$ to obtain a solution on $\mathbb{R}^N \times [0, 2T_*]$. An inductive argument completes the construction of the solution.

Properties iii) and iv) are true since they hold for the solutions of problem (P_T) which we constructed in Section 2. This proves the theorem. □

The next theorem states decay estimates for solutions of the Cauchy problem and growth estimates for their support.

Theorem 5.2. *Let the assumptions of Theorem* 5.1 *hold and let u be a solution to* (CP) *constructed as in Theorem* 5.1.

Then the following properties are satisfied:

i) *if $N = 2$ and $p \in (1, \infty)$ or $N = 3$ and $p \in (1, 6)$ then there exists a constant C depending on p, n, N such that for all $t > 0$*

$$|u(t)|_p \le C\, |u_0|_1^{\frac{4p+nN}{p(4+nN)}}\, t^{-\frac{p-1}{p}\frac{N}{4+nN}}, \tag{18}$$

ii) *there exists a constant* $C > 0$ *depending on* n, N *such that for all* $t > 0$

$$|\nabla u(t)|_2 \leq C\,|u_0|_1^{\frac{8+n(N-2)}{2(4+nN)}}\; t^{-\frac{1}{2}\frac{N+2}{4+nN}}, \tag{19}$$

iii) *if* $u_0(x) = 0$ *for all* $x \in \mathbb{R}^N$ *with* $|x| > R_0$, *then* $u(x,t) = 0$ *for all* $x \in \mathbb{R}^N$ *with* $|x| > R_0 + B\,|u_0|_1^{\frac{n}{4+nN}}\, t^{\frac{1}{4+nN}}$, *where* B *is a constant depending on* n *and* N.

Proof. As in Section 4 we introduce the function $w := u^{\frac{\alpha+n+1}{2}}$, and we derive from (17) that for all $t_1 < t_2 \in [0,\infty)$

$$\int_{\mathbb{R}^N} w^q(t_2) + c_1 \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |D^2 w|^2 \leq \int_{\mathbb{R}^N} w^q(t_1) \tag{20}$$

with $q = 2\frac{\alpha+1}{\alpha+n+1}$ and $\alpha \in (\max\{0, \frac{1}{2} - n\}, 2-n)$. From the Gagliardo–Nirenberg inequality in $\mathbb{R}^N$ (cf. [19]) and since mass is conserved ($\int_{\mathbb{R}^N} u(t) = \int_{\mathbb{R}^N} u_0$) we obtain, with $r = \frac{2}{\alpha+n+1}$ and a constant C depending only on N, q, r,

$$|w(t)|_q \leq C|D^2 w|_2^a |w|_r^{1-a} = C|D^2 w|_2^a |u_0|_1^{\frac{1-a}{r}}.$$

Here a is the exponent from the Gagliardo–Nirenberg inequality and is defined through $\frac{1}{q} = a(\frac{1}{2} - \frac{2}{N}) + (1-a)\frac{1}{r}$. We refer to Bernis ([3], Chapter 10) who demonstrates how to prove the Gagliardo–Nirenberg inequality if $0 < r < 1$.

Hence, (20) gives with $Y(t) := \int_{\mathbb{R}^N} w^q(t)$ that

$$Y(t_2) - Y(t_1) + C^{-1}|u_0|_1^{-\frac{2(1-a)}{ra}} \int_{t_1}^{t_2} Y^{\frac{2}{aq}}(\tau)d\tau \leq 0,$$

where C depends on α, n and N. This implies that

$$Y' \leq -C^{-1}|u_0|_1^{-\frac{2(1-a)}{ra}} Y^{\frac{2}{aq}}$$

in $D'(\mathbb{R}^+)$. Now combining this inequality with the $C([0,\infty); L^{\alpha+1}(\Omega))$-regularity of u (cf. Section 2) we observe that Y fulfills the assumptions of Lemma 6.5. Applying Lemma 6.5 with the choices $\Theta = \frac{2}{aq}$, $v_1 = Y$ and

$$v_2 = \left(Y_0^{1-\Theta} + C^{-1}|u_0|_1^{-\frac{2(1-a)}{ra}}(\Theta - 1)t\right)^{\frac{1}{1-\Theta}}$$

we can estimate

$$Y(t) \leq \left(Y_0^{-(\frac{2}{aq}-1)} + C^{-1}|u_0|_1^{-\frac{2(1-a)}{ra}}\left(\tfrac{2}{aq} - 1\right)t\right)^{\frac{aq}{aq-2}}.$$

Therefore, there exists a constant C depending on α, n and N such that

$$|u|_{\alpha+1} \leq C|u_0|_1^{\frac{4(\alpha+1)+nN}{(\alpha+1)(4+nN)}} t^{-\frac{\alpha}{\alpha+1}\frac{N}{4+nN}}$$

for $\alpha \in \big(\max\{0, \frac{1}{2}-n\}, 2-n\big)$. This proves (18) for $p \in \big(\max\{1, \frac{3}{2}-n\}, 3-n\big)$.

To derive the decay estimate for $|\nabla u(t)|_2$ we choose $\alpha = \frac{2-n}{2}$ and estimate

$$\int_{\mathbb{R}^N} |\nabla u|^2(t) = -\int_{\mathbb{R}^N} u\Delta u \leq \Big(\int_{\mathbb{R}^N} u^{\alpha+1}\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^N} u^{\alpha+n-1}|\Delta u|^2\Big)^{\frac{1}{2}}. \tag{21}$$

From the integral estimate (17) we derive

$$c_1 \int_t^{2t} \int_{\mathbb{R}^N} u^{\alpha+n-1}|\Delta u|^2 \leq Y(t).$$

Now we can use that $\int_{\mathbb{R}^N} |\nabla u(t)|^2$ is nonincreasing almost everywhere (cf. Theorem 5.1.iv) and inequality (21) to conclude that for almost all t

$$\begin{aligned} Y(t) &\geq c_1^{-1} \int_t^{2t} \int_{\mathbb{R}^N} u^{\alpha+n-1}|\Delta u|^2 \\ &\geq C^{-1} t \big(\sup_{s\in[t,2t]} \int_{\mathbb{R}^N} u^{\alpha+1}(s)\big)^{-1} \big(\int_{\mathbb{R}^N} |\nabla u|^2(2t)\big)^2, \end{aligned}$$

where C depends on n and N. This implies that for almost all t

$$\begin{aligned} \big(\int_{\mathbb{R}^N} |\nabla u|^2(2t)\big)^2 &\leq \big(\sup_{s\in[t,2t]} \int_{\mathbb{R}^N} u^{\alpha+1}(s)\big)^2 t^{-1} \\ &\leq C|u_0|_1^{\frac{2(4(\alpha+1)+nN)}{(4+nN)}} t^{-\frac{2\alpha N}{4+nN}-1} = C|u_0|_1^{\frac{2(8+n(N-2))}{(4+nN)}} t^{-\frac{2N+4}{4+nN}}. \end{aligned}$$

Since $t \mapsto \int_\Omega |\nabla u|^2(t)$ is lower semi-continuous, (19) is proved for all $t \in (0, \infty)$.

Having established (19) we are able to show the result on the decay behaviour for u in all L^p-norms with p as in i). The Gagliardo–Nirenberg inequality gives

$$|u|_p \leq C|\nabla u|_2^a \, |u_0|_1^{1-a}$$

with $a = \frac{2Np-2N}{(N+2)p}$. We remark that the assumptions on p and N ensure $a \in [0, 1]$. The decay estimate for the gradient then implies (18).

It remains to prove the growth estimate for the support of the solution u. Therefore, we make use of the scaling property that with u also

$$u_K(x, t) := K^N u(Kx, K^\gamma t), \quad \gamma := 4 + nN,$$

is a weak solution of the Cauchy problem having the same mass as u. The local integral estimate for u_K

$$\int_{\mathbb{R}^N} \zeta^4 u_K^{\alpha+1}(t_2) + C_1^{-1} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \zeta^4 |D^2 u_K^{\frac{\alpha+n+1}{2}}|^2$$
$$\leq \int_{\mathbb{R}^N} \zeta^4 u_K^{\alpha+1}(t_1) + C_2 \int_{t_1}^{t_2} \int_{[\zeta>0]} u_K^{\alpha+n-1}\big(|\nabla\zeta|^4 + \zeta^2|\Delta\zeta|^2\big)$$

where $\alpha \in \big(\max\{0, \frac{1}{2} - n\}, 2-n\big), \zeta \in C_0^2(\mathbb{R}^N)$
and C_1, C_2 are constants depending only on α and n,

follows by transformation of the local integral estimate for u (cf. Theorem 3.1 and Remark 3.2). The decay estimate i) gives

$$\begin{aligned}\int_{\mathbb{R}^N} u_K^{\alpha+1}(x,1)\,dx &= \int_{\mathbb{R}^N} K^{N(\alpha+1)} u^{\alpha+1}(Kx, K^\gamma)\,dx \\ &= \int_{\mathbb{R}^N} K^{N\alpha} u^{\alpha+1}(y, K^\gamma)\,dy \\ &\leq K^{N\alpha} C |u_0|_1^{\frac{4(\alpha+1)+nN}{4+nN}} K^{-\gamma\frac{\alpha N}{4+nN}} \leq C|u_0|_1^{\frac{4(\alpha+1)+nN}{4+nN}}.\end{aligned}$$

Now we apply Theorem 4.2 to the functions u_K at time $t_0 = 1$. The above estimate for $\int_{\mathbb{R}^N} u_K^{\alpha+1}(x,1)\,dx$ gives that T_* in Theorem 4.2 only depends on α, n, N, R_0 and $|u_0|_1$ (cf. Remark 4.3). A covering argument as in the proof of Theorem 5.1 gives that for all $\bar{R} > 0$ there exists a time $\bar{T} > 1$ depending on $\alpha, n, N, \bar{R}$ and $|u_0|_1$ such that for all $A \in \mathbb{R}^+$ the following property holds:

$$\begin{aligned}&\text{if} \quad u_K(x,1) = 0 \text{ for all } x \in \mathbb{R}^N \text{ with } |x| > A, \\ &\text{then}\, u_K(x,t) = 0 \text{ for all } x \in \mathbb{R}^N \text{ with } |x| > A + \bar{R} \text{ and } t \in [1, \bar{T}].\end{aligned} \tag{A_K}$$

When transformed back to u, property (A_K) implies that for all $\tau_0 > 0$ and all $\bar{A} \in \mathbb{R}^+$

$$\begin{aligned}&\text{if} \quad u(x,\tau_0) = 0 \text{ for all } x \in \mathbb{R}^N \text{ with } |x| > \bar{A}, \\ &\text{then} \quad u(x,t) = 0 \text{ for all } x \in \mathbb{R}^N \text{ with } |x| > \bar{A} + \bar{R}\,\tau_0^{1/\gamma} \text{ and } t \in [\tau_0, \tau_0\bar{T}].\end{aligned} \tag{A}$$

By assumption we have $u_0(x) = 0$ if $|x| > R_0$. Theorem 4.2 and a covering argument as in the proof of Theorem 5.1 give the existence of a time $T_* > 0$ and a function

$$R : [0, T_*] \to \mathbb{R}^+, \quad R(0) = R_0$$

such that

$$u(x,t) = 0 \quad \text{if} \quad |x| > R(t) \quad \text{and} \quad t \in [0, T_*].$$

Now we fix a time $T_0 \in (0, T_*)$, and we apply property (A) with $\tau_0 = T_0$ to derive

$$u(x,t) = 0 \text{ if } |x| > R(T_0) + \bar{R}\, T_0^{1/\gamma} \text{ and } t \in [T_0, T_0 \bar{T}].$$

If we apply property (A) again, this time setting $\tau_0 = T_0 \bar{T}$ and $\bar{A} = R(T_0) + \bar{R}\, T_0^{1/\gamma}$ we get

$$u(x,t) = 0 \text{ if } |x| > R(T_0) + \bar{R}\, T_0^{1/\gamma}(1 + \bar{T}^{1/\gamma}) \text{ and } t \in [T_0 \bar{T}, T_0\, (\bar{T})^2].$$

Induction gives for $k \geq 1$

$$u(x,t) = 0 \text{ if } |x| > R(T_0) + \bar{R}\, T_0^{1/\gamma} \sum_{j=0}^{k-1} (\bar{T})^{j/\gamma} \text{ and } t \in [T_0\, (\bar{T})^{k-1}, T_0\, (\bar{T})^k].$$

Hence for $k \geq 2$ it holds that

$$u(x,t) = 0 \text{ if } |x| > R(T_0) + \bar{R}\, T_0^{1/\gamma}\, \frac{(\bar{T})^{k/\gamma} - 1}{(\bar{T})^{1/\gamma} - 1} \text{ and } t \in [T_0\, (\bar{T})^{k-1}, T_0\, (\bar{T})^k].$$

If $t \in [T_0\, (\bar{T})^{k-1}, T_0\, (\bar{T})^k]$ we estimate

$$\bar{R}\, T_0^{1/\gamma}\, \frac{(\bar{T})^{k/\gamma} - 1}{(\bar{T})^{1/\gamma} - 1} \leq \frac{\bar{R}\, (\bar{T})^{1/\gamma}}{(\bar{T})^{1/\gamma} - 1}\, t^{1/\gamma},$$

and hence

$$u(x,t) = 0 \text{ if } |x| > R(T_0) + \bar{B}\, t^{1/\gamma} \text{ and } t > T_0\, \bar{T}, \tag{22}$$

where the constant $\bar{B}$ depends on n, N and $|u_0|_1$. Using (14), near $t = 0$ $R(t)$ can be estimated in the following way:

$$R(t) \leq R_0 + r_0 - \sup_{R_0 + r_0} \left(r_0^N - A_0 t^{\eta/4} \Big(\int_0^t \int_{B_{r_0}(x_0)} |D^2 u^{\frac{\alpha+n+1}{2}}|^2 \, dx \, dt \Big)^{\beta} \right)^{\frac{1}{N}}.$$

Now (17) implies that for $t \to 0$ the right-hand side converges to R_0; i.e., in relation (22) $R(T_0)$ can be replaced by R_0, and thus (22) holds for all $t > 0$.

Finally, we use that the local integral estimate is invariant under the scaling

$$u_M(x,t) = M u(x, M^n t) \tag{23}$$

to conclude that $\bar{B}$ in (22) can be chosen as

$$\bar{B} = B |u_0|_1^{\frac{n}{4+nN}}$$

with a constant B depending only on n and N but not on u_0. This can be seen by the following argument. Because of the scaling property (23) we only need to show the growth estimate for initial data with mass one. The form of $\bar{B}$ then follows by scaling solutions with arbitrary mass to a solution with mass one. This completes the proof of the theorem.

Remark. i) In one space dimension the method to prove Theorem 5.2 would give the results of Bernis (cf. Theorem 7.1 and inequality (7.7) of [3]).

ii) The powers in the decay estimates of Theorem 5.2 are optimal, as can be seen from self-similar source-type solutions of the form

$$u(x,t) = t^{-N\lambda}\,\hat{u}\left(\frac{|x|}{t^{\lambda}}\right) \text{ where } \lambda = \frac{1}{4+nN}$$

which have been found by Bernis and Ferreira ([12]). A self-similar source-type solution therefore necessarily has the decay $t^{-\frac{p-1}{p}\frac{N}{4+nN}}$ for the L^p-norm, $t^{-\frac{1}{2}\frac{N+2}{4+nN}}$ for $|\nabla u(t)|_2$ and a growth rate of $t^{\frac{1}{4+nN}}$ for its support.

iii) Scaling arguments are widely used in the literature, but the combination with the induction argument leading to the proof of (22) seems to be new.

6. Auxiliary inequalities. In this section we prove some auxiliary inequalities used in Sections 4 and 5. We introduce the notation

$$|v|_{p,r}^p := \int_{B_r(0)} |v|^p \text{ for all } p \in (0,\infty).$$

First we state a version of the Gagliardo–Nirenberg inequality appropriate for our purposes. In particular we state the dependence on the radius r explicitly.

Lemma 6.1. (Gagliardo–Nirenberg) *Let $0 < q < 2$. Then there exist constants K_1 and $\hat{K}_1$ such that for all $r > 0$ and $v \in H^2\big(B_r(0)\big)$*

$$|v|_{2,r} \leq K_1 |D^2 v|_{2,r}^a |v|_{q,r}^{1-a} + \hat{K}_1 r^{-\mu} |v|_{q,r}, \tag{24}$$

where

$$a = \frac{(2-q)N}{(4-N)q+2N} \quad \text{and} \quad \mu = \frac{(2-q)N}{2q}.$$

If $B_r(0)$ is replaced by $\mathbb{R}^N$, then (24) holds with the constant $\hat{K}_1 = 0$.

The Gagliardo–Nirenberg inequality was independently proved by Gagliardo ([13]) and Nirenberg ([18, 19]). Bernis proved the lemma for the nonstandard case $q \in (0,1)$ in one space dimension. The generalization to higher space dimensions is straightforward. The explicit dependence of the constants on r follows by a simple scaling argument.

The following lemma generalizes Lemma 10.5 of Bernis [3] to higher space dimensions.

Lemma 6.2. *Let $0 < q < 2$. Then there exists a constant K_2 such that for all $r > 0$ and all $v \in L^2(B_r(0))$*

$$\int_{B_r(0)} |v|^q \le K_2 \Big(\int_{B_r(0)} \big(r^N - |x|^N\big)^4 v^q \Big)^c \Big(\int_{B_r(0)} v^2 \Big)^{\frac{q}{2}(1-c)},$$

where $c = \frac{2-q}{10-q}$.

Proof. For $s \in (0, r)$ we get

$$\begin{aligned}
&\int_{B_r(0)} v^q = \int_{B_s(0)} v^q + \int_{B_r(0)\setminus B_s(0)} v^q \\
&\le \big(r^N - s^N\big)^{-4} \int_{B_s(0)} \big(r^N - |x|^N\big)^4 v^q + \Big(\int_{B_r(0)\setminus B_s(0)} v^2 \Big)^{\frac{q}{2}} C_{N,q} \big(r^N - s^N\big)^{\frac{2-q}{2}} \\
&\le \big(r^N - s^N\big)^{-4} \int_{B_r(0)} \big(r^N - |x|^N\big)^4 v^q + \Big(\int_{B_r(0)} v^2 \Big)^{\frac{q}{2}} C_{N,q} \big(r^N - s^N\big)^{\frac{2-q}{2}}.
\end{aligned}$$

Now we minimize the function $f(z) = az^{-4} + bz^{\frac{2-q}{2}}$ on the interval $(0, r^N]$, where we set $a = \int_{B_r(0)} (r^N - |x|^N)^4 v^q$ and $b = (\int_{B_r(0)} v^2)^{\frac{q}{2}} C_{N,q}$. The point $z_0 := (\frac{8a}{(2-q)b})^{\frac{2}{10-q}}$ is the only positive number with $f'(z) = 0$. It follows that either f has a minimum $z_0 \in (0, r^N]$ or $z_0 > r^N$. Both possibilities imply the inequality stated in the lemma. $\square$

The next lemma supplies an auxiliary inequality, which was used in Section 4 to show finite speed of propagation.

Lemma 6.3. *Let $0 < q < 2$. Then there exists a constant K_3 such that for all $r > 0$ and all $v \in H^2\big(B_r(0)\big)$*

$$\begin{aligned}
K_3 |v|_{2,r}^2 \le |D^2 v|_{2,r}^{2d} \Big(\int_{B_r(0)} \big(r^N - |x|^N\big)^4 v^q \Big)^{\frac{2}{q}(1-d)} \\
+ r^{-2\nu} \Big(\int_{B_r(0)} \big(r^N - |x|^N\big)^4 |v|^q \Big)^{\frac{2}{q}},
\end{aligned}$$

where we have set

$$d = \frac{(10-q)N}{10N + (4-N)q} \quad \textit{and} \quad \nu = \frac{(10-q)N}{2q}.$$

Proof. The Lemmas 6.1 and 6.2 yield

$$\begin{aligned}
|v|_{2,r} &\le K_1 |D^2 v|_{2,r}^a |v|_{q,r}^{1-a} + \hat{K}_1 r^{-\mu} |v|_{q,r} \\
&\le C |D^2 v|_{2,r}^a \Big(\int_{B_r(0)} \big(r^N - |x|^N\big)^4 |v|^q \Big)^{\frac{c}{q}(1-a)} |v|_{2,r}^{(1-c)(1-a)} \\
&\quad + C r^{-\mu} \Big(\int_{B_r(0)} \big(r^N - |x|^N\big)^4 |v|^q \Big)^{\frac{c}{q}} |v|_{2,r}^{1-c}.
\end{aligned}$$

Dividing by $|v|_{2,r}^{(1-c)(1-a)}$, using Young's inequality for the second term on the right-hand side and taking the power $\frac{2}{a(1-c)+c}$, we obtain

$$|v|_{2,r}^2 \le C|D^2 v|_{2,r}^{\frac{2a}{a(1-c)+c}} \Big(\int_{B_r(0)} \big(r^N - |x|^N\big)^4 |v|^q\Big)^{\frac{c(1-a)}{a(1-c)+c}\frac{2}{q}}$$
$$+ Cr^{-\frac{2\mu}{c}} \Big(\int_{B_r(0)} \big(r^N - |x|^N\big)^4 |v|^q\Big)^{\frac{2}{q}}.$$

Now the claim follows from the definition of a and c. □

Let

$$R_s(r) := \int_{B_r(0)} \big(r^N - |x|^N\big)^s \, \Phi(x)\, dx$$

with

$$\Phi \in L^1\big(B_{r_0}(0)\big), \quad \Phi \ge 0, \quad r_0 > 0.$$

Assume

$$R_4(r) \le K\, R_0^{\theta}(r)$$

for $r \in [r_m, r_0]$. Then the following lemma holds (cf. Lemma 11.1 of [3]).

Lemma 6.4. *Let $K > 0$, $\theta > 1$ and $0 \le r_m < r_0$. If for the real number r_1 defined through*

$$r_1^N = r_0^N - \tfrac{\theta+3}{\theta-1} K^{\frac{1}{4}} R_0(r_0)^{\frac{\theta-1}{4}}$$

the inequality $r_1 \ge \max\{0, r_m\}$ holds, then $R_0(r) = 0$ for $r \in (0, r_1]$.

This lemma is proved in the same way as Lemma 11.1 in Bernis [3]. We only need to use that $R_1' = N r^{N-1} R_0$. But we might as well switch to the variable $z = r^N$ and apply Lemma 11.1 directly.

Lemma 6.5. *Let $\Theta > 1$, $C > 0$, $M > 0$ and assume that the functions $v_1, v_2 : [0,\infty) \to (0, M)$ are continuous and have the following properties:*

i) *v_1 is monotonically decreasing and fulfills ${v_1}' \le -Cv_1^{\Theta}$ in $D'(\mathbb{R}^+)$;*
ii) *${v_2}' \ge -Cv_2^{\Theta}$ in $D'(\mathbb{R}^+)$;*
iii) *$v_2(0) \ge v_1(0)$.*

Then $v_2(t) \ge v_1(t)$ for all $t \in \mathbb{R}^+$.

Proof. The function $s : [0,\infty) \to \mathbb{R}_0^+$, defined by

$$s(t) := \begin{cases} C\frac{v_2(t)^{\Theta} - v_1(t)^{\Theta}}{v_2(t) - v_1(t)} & \text{if } v_2(t) \ne v_1(t), \\ \Theta C v_2(t)^{\Theta-1} & \text{if } v_2(t) = v_1(t) \end{cases}$$

is bounded and nonnegative. For arbitrary $t_2 > 0$ and an arbitrary, nonnegative function $\Psi \in C_0^\infty((0,t_2))$ we have

$$-\int_0^{t_2} \Psi'(t)(v_2(t) - v_1(t))\,dt \geq -C \int_0^{t_2} (v_2(t)^\Theta - v_1(t)^\Theta)\Psi(t)\,dt. \qquad (25)$$

Now we approximate an arbitrary nonnegative Lipschitz functions $\phi \in C^{0,1}([0,t_2])$ by smooth functions with compact support. Hence, inequality (25), the continuity of v_1, v_2 and item iii) imply

$$(v_2(t_2) - v_1(t_2))\phi(t_2) \geq \int_0^{t_2} (v_2(t) - v_1(t))(\phi'(t) - s(t)\phi(t))\,dt.$$

Choosing ϕ as the solution of the ordinary differential equation $\dot{K}(t) = s(t)K(t)$, $K(0) = K_0 > 0$ and using that ϕ is positive, the result can be established.

7. Appendix. We are left to prove the estimates (10). We formulate the result in the following lemma.

Lemmar 7.1. *There exist constants C (independent of σ and δ) and $\hat{C} = \hat{C}(\sigma,\delta)$ such that for all $\tau \geq 0$ and for all $\varepsilon > 0$*

$$\begin{aligned} m'_{\sigma\delta}(\tau)\, g^\alpha_{\sigma\delta}(\tau+\varepsilon) &\leq C(\tau+\varepsilon)^{\alpha+n-1} + \hat{C}(\sigma,\delta)\,\varepsilon\,(\tau+\varepsilon)^{\alpha+n-1}, \\ m_{\sigma\delta}(\tau)\,(g^\alpha_{\sigma\delta})'(\tau+\varepsilon) &\leq C(\tau+\varepsilon)^{\alpha+n-1}, \\ m_{\sigma\delta}(\tau)\, g^\alpha_{\sigma\delta}(\tau+\varepsilon) &\leq C(\tau+\varepsilon)^{\alpha+n}. \end{aligned}$$

Proof. We only prove the first inequality. The second and the third inequalities are easier to establish since one can use that $m_{\sigma\delta}(\tau) \leq m_{\sigma\delta}(\tau+\varepsilon)$, and therefore we omit the proofs.

We define Z, N, P_1, P_2, P_3 such that

$$g^\alpha_{\sigma\delta}(\tau) = \tfrac{\delta}{\alpha+n-s}\tau^{\alpha+n-s} + \tfrac{1}{\alpha}\tau^\alpha + \tfrac{\sigma}{\alpha+n}\tau^{\alpha+n} =: P_1(\tau) + P_2(\tau) + P_3(\tau)$$

and

$$m'_{\sigma\delta}(\tau) = \frac{s\delta\tau^{s+2n-1} + n\tau^{2s+n-1}}{(\delta\tau^n + \tau^s + \sigma\tau^{s+n})^2} =: \frac{Z(\tau)}{N(\tau)}.$$

Now we estimate

$$\begin{aligned} P_1(\tau+\varepsilon)\,Z(\tau) &= (\tau+\varepsilon)^{\alpha+n-1}\big(\tfrac{\tau+\varepsilon}{\tau}\big)^{1-s}\big[\tfrac{s\delta^2}{\alpha+n-s}\tau^{2n} + \tfrac{\delta n}{\alpha+n-s}\tau^{n+s}\big] \\ &\leq (\tau+\varepsilon)^{\alpha+n-1}\big[\tfrac{s\delta^2}{\alpha+n-s}\tau^{2n} + \tfrac{\delta n}{\alpha+n-s}\tau^{n+s}\big] \end{aligned}$$

and

$$P_2(\tau+\varepsilon)\,Z(\tau) = (\tau+\varepsilon)^{\alpha+n-1}\left(\tfrac{\tau+\varepsilon}{\tau}\right)^{1-n}\left[\tfrac{s\delta}{\alpha}\tau^{s+n}+\tfrac{n}{\alpha}\tau^{2s}\right]$$
$$\leq \begin{cases} (\tau+\varepsilon)^{\alpha+n-1}\left[\tfrac{s\delta}{\alpha}\tau^{s+n}+\tfrac{n}{\alpha}\tau^{2s}\right] & \text{if } n\geq 1,\\ (\tau+\varepsilon)^{\alpha+n-1}(1+\tfrac{\varepsilon}{\tau})\left[\tfrac{s\delta}{\alpha}\tau^{s+n}+\tfrac{n}{\alpha}\tau^{2s}\right] & \text{if } n<1. \end{cases}$$

For the last inequality we used that $(1+\frac{\varepsilon}{\tau})^{1-n}\leq 1+\frac{\varepsilon}{\tau}$ if $n<1$.

In addition we get

$$P_3(\tau+\varepsilon)\,Z(\tau) = (\tau+\varepsilon)^{\alpha+n-1}\left(\tfrac{\tau+\varepsilon}{\tau}\right)\left[\tfrac{s\sigma\delta}{\alpha+n}\tau^{s+2n}+n\sigma\tau^{2s+n}\right].$$

This gives

$$\begin{aligned} &m'_{\sigma\delta}(\tau)g^{\alpha}_{\sigma\delta}(\tau+\varepsilon)\\ &\leq C(\tau+\varepsilon)^{\alpha+n-1}\left\{\tfrac{\delta^2\tau^{2n}+\delta\tau^{n+s}+\tau^{2s}+\sigma\delta\tau^{s+2n}+\sigma\tau^{2s+n}}{\delta^2\tau^{2n}+\delta\tau^{n+s}+\sigma\delta\tau^{s+2n}+\tau^{2s}+\sigma\tau^{2s+n}+\sigma^2\tau^{2s+2n}}\right\}\\ &+C\varepsilon(\tau+\varepsilon)^{\alpha+n-1}\left[\tfrac{\delta\tau^{n+s-1}+\tau^{2s-1}+\sigma\delta\tau^{s+2n-1}+\sigma\tau^{2s+n-1}}{\delta^2\tau^{2n}+\delta\tau^{n+s}+\sigma\delta\tau^{s+2n}+\tau^{2s}+\sigma\tau^{2s+n}+\sigma^2\tau^{2s+2n}}\right]. \end{aligned}$$

Now we can estimate the factor $\{\quad\}$ by a constant, which does not depend on δ,σ and τ. The factor $[\quad]$ is for all $\tau\geq 0$ bounded by a constant depending on σ and δ. This proves Lemma 6.1.

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