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A multi-phase Mullins–Sekerka system: matched asymptotic expansions and an implicit time discretisation for the geometric evolution problem

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We propose a generalisation of the Mullins–Sekerka problem to model phase separation in multi-component systems. The model includes equilibrium equations in bulk, the Gibbs–Thomson relation on the interfaces, Young's law at triple junctions, together with a dynamic law of Stefan type. Using formal asymptotic expansions, we establish the relationship to a transition layer model known as the Cahn–Hilliard system. We introduce a notion of weak solutions for this sharp interface model based on integration by parts on manifolds, together with measure theoretical tools. Through an implicit time discretisation, we construct approximate solutions by stepwise minimisation. Under the assumption that there is no loss of area as the time step tends to zero, we show the existence of a weak solution.

1. Introduction

In this paper, we study some mathematical models describing a chemical system of N species undergoing phase transition under isothermal conditions. Our starting point is a Cahn–Hilliard system which contains a small length scale parameter ε. We then formally derive a limiting system of equations as $\varepsilon \to 0$. This leads us to propose a multi-phase analogue of the Mullins–Sekerka problem in $\mathbf{R}^3$ including triple junction lines and quadruple junctions. We present a weak formulation for this system and prove a conditional existence result using an implicit time discretisation approach.

The Cahn–Hilliard system we consider takes the form

$$\text{(I)} \qquad \partial_t u = \Delta w,$$

$$\text{(II)} \qquad w = -\varepsilon \Delta u + \frac{1}{\varepsilon} f(u),$$

where the spatial variable ranges in some open, bounded $\Omega \subset \mathbf{R}^3$ and where $u:(0, T) \times \Omega \to R^N$ ($N \geqq 3$) represents the vector of molar functions or of mass densities of the components in the chemical system, and $w:(0, T) \times \Omega \to R^N$ is the associated vector of chemical potentials.

Since u_j is the concentration of the jth component of the chemical system, it should be non-negative and the sum of all u_j's should be one. Thus we constrain u to lie in the so-called Gibbs triangle, defined by $\{u \in \mathbf{R}^N : \Sigma_{i=1}^N u_i = 1, u_i \geqq 0\}$. Now, let e be the vector $(1, \ldots, 1)$ in $\mathbf{R}^N$. In view of the constraint, we introduce the hyperplane

$$\Sigma = \{u \in \mathbf{R}^N \mid u \cdot e = 1\},$$

and its tangent plane

$$T\Sigma = \{u \in \mathbf{R}^N \mid u \cdot e = 0\}.$$

Next, let $\psi : \mathbf{R}^N \to \mathbf{R}$ be a potential with an N-well structure on Σ, i.e. let $\psi_{|\Sigma}$ have exactly N local minima, which are of equal height. We define

$$f := \nabla_u \psi - \frac{1}{N} (\nabla_u \psi \cdot e) e$$

to be the projection of $\nabla_u \psi$ onto the tangent plane $T\Sigma$ and we set $B := \nabla_u f$. We assume that B is positive definite at the minima. The free energy is then given by

$$E[u] := \int_\Omega \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} \psi(u) \, dx$$

for functions $u : \Omega \to \Sigma$. The vector $w : \Omega \to T\Sigma$ of chemical potentials is the gradient of $E[u]$ with respect to the L^2 inner product. Thus its value is given by (II). Equation (I) is a balance law which is a consequence of mass conservation (cf. [17]).

We study (I)–(II) subject to the natural boundary conditions

$$\frac{\partial}{\partial \nu} u = 0, \quad \frac{\partial}{\partial \nu} w = 0$$

on $\partial\Omega$.

Elliott and Luckhaus [17] have proved the existence of solutions for this problem. We refer to recent work of Elliott and Garcke [16] for the case that a nonconstant mobility matrix is introduced in the model.

In this paper, we concentrate on the case $N \geqq 3$ and we study asymptotic expansions of solutions of the system (I)–(II), provided ε is small. A similar study has been carried out by Bronsard and Reitich [5] for the Allen–Cahn system

$$\varepsilon u_t - \varepsilon \Delta u + \frac{1}{\varepsilon} \nabla_u \psi(u) = 0,$$

where the vector u of order parameters is not conserved. They obtained formally that each interface evolves by its mean curvature and that an angle condition must be satisfied at triple junctions. Then they proved that this limiting system of partial differential equations is well posed in the classical sense by combining $C^{2,\alpha}$-Schauder estimates with a fixed-point argument. For the Cahn–Hilliard system (I)–(II), the formal asymptotic expansion and the study of the limiting problem are harder, since the mass conservation of the individual species introduces a nonlocal aspect to this system of equations.

Using the results of the asymptotic expansion, we derive a multi-phase analogue of the Mullins–Sekerka problem (see Section 2). Roughly speaking, we can write this system around a (time-dependent) triple junction line as:

- $\Delta w = 0$ in each bulk region;
- $[w]_\Gamma = 0$ on Γ;
- $[\nabla_x w]_\Gamma \cdot \nu_\Gamma = -v_\Gamma [u]_\Gamma$ on Γ;
- $w \cdot [u]_\Gamma = -\sigma\kappa_\Gamma$ on Γ;
- the angles are 120° between the three interfaces (Young's law).

Here Γ denotes any one of the three surfaces meeting at the triple junction line, $[\cdot]_\Gamma$ denotes the jump across Γ, ν_Γ is the normal, v_Γ the normal velocity, κ_Γ is the sum of principal curvatures of Γ, while σ is the surface tension, being a given constant. In other words, we obtain a geometrical model in which the evolution of the surfaces separating the bulk regions is determined through this system of partial differential equations. We present a precise formulation of this geometrical problem under the assumption that there are no singularities such as coalescing quadruple junctions or self-intersecting interfaces (see Definition 3.1). We do this while allowing the surface tension σ to be different on different interfaces, giving rise to a modification of Young's law. We also derive some geometrical properties of this flow (see Section 3, third subsection).

In the binary case ($N = 2$), the associated formal result was obtained by Pego [**31**]. The proof of the convergence to the Mullins–Sekerka problem was given in the radial case by Stoth [**36**] and in the general domain case by Alikakos, Bates and Chen [**1**] assuming smoothness of the interfaces and particular initial data. Recently Chen [**10**] adapted the geometric measure theoretical approach of Ilmanen [**25**] (see also [**34**]) to prove the convergence result in the binary case without the assumption of smoothness of the interfaces. Many of the methods used in these convergence results are inspired by those which were developed to prove the convergence of the Allen–Cahn equation to the mean curvature flow of the interfaces (see [**13, 20**]). We note that no rigorous convergence results have been obtained yet for the Allen–Cahn system. So we expect serious difficulties in the Cahn–Hilliard system.

The limiting system above can be interpreted as a multi-phase analogue of the Mullins–Sekerka problem. The classical existence result for the Mullins–Sekerka problem has been obtained recently by Escher and Simonett [**18**] and independently by Chen, Hong and Yi [**12**]. They used the method developed by Duchon and Robert [**14**] for the case when the interface is a graph over **R**. It is based on integral representation formulae and fixed-point arguments. The new features of the present system are the triple junctions lines and the angle conditions, which present obstacles to this approach.

Instead, we extend the weak formulation approach developed by Luckhaus [**26**] in his study of the Stefan problem with the Gibbs–Thomson law (see Definition 4.1). The main tool of this formulation is a weak form of the curvature equation, using integration by parts on manifolds. We use conservation of mass to overcome the difficulty that chemical potentials are only defined up to a constant, so that we are able to treat the Neumann problem. We show that this weak formulation is consistent with the strong formulation (see Proposition 4.3). In particular, we point out that the angle condition arises as the natural boundary condition when performing the integration by parts of the curvature equation. We prove a *conditional* existence result for the multi-phase Mullins–Sekerka problem (see Theorem 5.8). The method we use is based on implicit discretisation in time, ensuring the existence of solutions at each time step via a variational formulation. We use *a-priori* estimates to pass to the continuous model. This result is conditional in nature, since we have to impose a convergence condition on the semidiscretised approximation that we are not able to verify. This condition says that there is no loss of perimeter in the limit as the time step converges to zero.

This time discrete approach was successful in developing a weak formulation of the mean curvature flow for general surfaces with singularities such as junctions (see [**2**, **27**]). It has the advantage that it does not allow fattening of the interfaces (as opposed to the viscosity solution approach of Chen, Giga and Goto [**11**] and Evans and Spruck [**19**]). Hence coupling with 'bulk' equations is possible. Luckhaus and Sturzenhecker [**27**] also used this weak formulation for the case of a two-phase Mullins–Sekerka problem. We extend their work to the $N \geqq 3$-dimensional case and the case of Neumann boundary data, and incorporate the triple junction conditions in this weak framework.

We conclude with some remarks on studies related to ours. Cahn and Novick-Cohen [**8**] derive an Allen–Cahn/Cahn–Hilliard system to describe simultaneous phase separation and ordering in binary alloys. In their model, the mobility depends on the order parameter and degenerates at special values. Applying formal asymptotic expansions, they derive that on a small timescale the limiting motion consists in a certain coupling of motion by mean curvature and motion by minus the surface Laplacian [**9**, **28**].

Garcke and Novick-Cohen [**21**] consider a Cahn–Hilliard system with degenerate mobility (cf. [**16**]). They obtain formally that in the singular limit the interfaces move according to motion by surface diffusion. This result has already been derived in the binary case by Cahn, Elliott and Novick-Cohen [**7**]. The new feature is a balance of fluxes, which is an independent condition at the triple point, and has to be incorporated in the model in addition to the continuity of the chemical potentials and Young's law.

2. Formal asymptotics

In this section, we perform a formal asymptotic analysis of the vector-valued Cahn–Hilliard system, assuming that ε is small. We do not intend to derive all terms of the expansion. Our goal is to expand the solution up to the order necessary to obtain a well-posed limit problem.

We denote the N minima of $\psi_{|\Sigma}$ by $A(j)$ $(j = 1, \ldots, N)$, with height $\psi(A(j)) = 0$. This latter condition is assumed to make integration constants vanish, that otherwise appeared at several places in the formal expansions.

A prototype example is given by the logarithmic potential

$$\psi(u) = \theta \sum_{i=1}^{N} u_i \ln u_i - \frac{1}{2} u \cdot (Id - ee^T) u + c(\theta),$$

where θ denotes the absolute temperature, which is treated as a fixed constant here, and with $c(\theta)$ chosen such that $\psi(A(j))=0$. The projection of $\nabla_u\psi(u)$ onto $e^\perp$ is given by

$$f(u)=\frac{1}{N}(e\cdot u)e-u+\theta\left(\ln u-\frac{1}{N}(\ln u\cdot e)e\right).$$

If θ is sufficiently small, we note that ψ has an N-well structure on the hyperplane Σ. The minima are located at the N points given by the formula $A(j):=\delta e+(1-N\delta)e_j$, and f vanishes there. Here $\delta=\delta(\theta)$ is some well-defined function with $\delta(\theta)\leqq Ce^{-1/\theta}$ as $\theta\to 0$. Thus in the deep quench limit ($\theta\to 0$) the minima approach the corners of the Gibbs triangle. We note that the matrix $B(u)$ is given by

$$B_{ij}(u)=f_{i,u_j}(u)=\frac{1}{N}-\delta_{ij}-\frac{1}{N}\frac{\theta}{u_j}+\delta_{ij}\frac{\theta}{u_i}.$$

By construction, $B(u)$ maps the tangent plane $T\Sigma$ onto itself. Since θ/δ converges to ∞ as $\theta\to\infty$, we conclude that $B(A(j))$ is positive definite on $T\Sigma$ for small positive values of θ. At some places we will point out how identities simplify in the case of this prototype potential.

Outer expansion

We make the ansatz

$$u_{\text{out}}(t,x;\varepsilon)=\sum_{K=0}^{\infty}\varepsilon^K u^K(t,x),\quad w_{\text{out}}(t,x;\varepsilon)=\sum_{K=0}^{\infty}\varepsilon^K w^K(t,x),$$

with $u^0\in\Sigma$ and $u^{K\geqq 1}$, $w^{K\geqq 2}\in T\Sigma$. We substitute this ansatz into the Cahn–Hilliard system.

The $\mathcal{O}(1/\varepsilon)$-equation of (II) is

$$f(u^0)=0.$$

The stable solutions of this equation are $u^0=A(j)$ for some $j\in\{1,\ldots,N\}$.

The $\mathcal{O}(1)$-equation of (I) is

$$\partial_t u^0=\Delta w^0.$$

Since u^0 is a constant, this implies $\Delta w^0=0$.

Inner expansion in the transition layer

We construct a solution that makes a transition from some $A(j)$ to some different $A(k)$ across some smoothly evolving manifold $\Gamma(t)=\Gamma(t;0)$. Assume that $\Gamma(t;\varepsilon)$ is a smoothly evolving manifold, with spatial normal $\nu(t,\cdot;\varepsilon)$. We introduce coordinates $r(t,\cdot;\varepsilon)$ and $s(t,\cdot;\varepsilon)$ in some suitable neighbourhood of $\Gamma(t;\varepsilon)$:

We set $r(t,\cdot;\varepsilon)$ to be the signed distance function, positive in the direction of $\nu(t,\cdot;\varepsilon)$. In the neighbourhood of $\Gamma(t;\varepsilon)$ the projection $P(t;\varepsilon)$ onto $\Gamma(t;\varepsilon)$ is well defined. We assume that $\Gamma(t;\varepsilon)$ is locally diffeomorphic to $\Gamma(t;0)$ with associated diffeomorphism $\Phi(t,\cdot;\varepsilon)$ and we define $s(t,x;\varepsilon)$ through $\Phi(t,s(t,x;\varepsilon);\varepsilon)=P(t;\varepsilon)x$.

We rescale the r-variable to $z(t,x;\varepsilon)=(1/\varepsilon)r(t,x;\varepsilon)$.

The pioneering mathematical work which used this parametrisation in a systematic way was done by De Mottoni and Schatzman [**13**].

We make the ansatz

$$u_{\text{in}}(t, x; \varepsilon) = \sum_{K=0}^{\infty} \varepsilon^K U^K(t, z(t, x; \varepsilon), s(t, x; \varepsilon)),$$

$$w_{\text{in}}(t, x; \varepsilon) = \sum_{K=0}^{\infty} \varepsilon^K W^K(t, z(t, x; \varepsilon), s(t, x; \varepsilon)),$$

with $U^0 \in \Sigma$ and $U^{K \geqq 1}$, $W^{K \geqq 0} \in T\Sigma$. We substitute this ansatz into the Cahn–Hilliard system.

The $\mathcal{O}(1/\varepsilon^2)$-equation of (I) reads

$$\partial_{zz} W^0 = 0.$$

We are looking for bounded W^0, and thus $W^0(t, z, s) = W^0(t, s)$.

The $\mathcal{O}(1/\varepsilon)$-equation of (II) is

$$0 = -\partial_{zz} U^0 + f(U^0).$$

Since f is the projection of $\nabla_u \psi$ onto the plane $e^\perp$, there exists a solution $\bar{U}$ of this system of ODEs satisfying the constraint $\bar{U} \cdot e = 1$ and connecting the values $A(j)$ at $z = -\infty$ and $A(k)$ at $z = +\infty$. For this existence we refer to [35]. In particular, Sternberg shows that

$$\sigma = \sigma_{jk} := \int_{-\infty}^{+\infty} |\partial_z \bar{U}|^2 \, dz = 2 \int_{-\infty}^{+\infty} \psi(\bar{U}) \, dz$$

is finite. We set $U^0(t, z, s) := \bar{U}(z)$.

The $\mathcal{O}(1/\varepsilon)$-equation of (I) is

$$(\partial_t r)^0 \partial_z U^0 = \partial_{zz} W^1,$$

where $(\partial_t r)^0(t, z, s) = \partial_t r(t, x_\varepsilon; \varepsilon)_{|\varepsilon=0}$ with $x_\varepsilon = \Phi(t, s; \varepsilon) + \varepsilon v(t, \Phi_{jk}(t, s; \varepsilon); \varepsilon) z$. Since $(\partial_t r)^0$ is independent of z, we may integrate this with respect to z and obtain

$$(\partial_t r)^0 U^0 |_{z=-\infty}^{z=+\infty} = \partial_z W^1 |_{z=-\infty}^{z=+\infty}.$$

We note that $(\partial_t r)^0 = -v$, where v is the normal velocity v of $\Gamma(t; 0)$. In the case of the prototype potential ψ, the above identity may be restated as $-v(1 - N\delta) = \partial_z W_k^1 |_{-\infty}^{+\infty} = -\partial_z W_j^1 |_{-\infty}^{+\infty}$, and $0 = \partial_z W_i^1 |_{-\infty}^{+\infty}$ for $i \neq j, k$.

The $\mathcal{O}(1)$-equation of (II) is

$$W^0 = -\partial_{zz} U^1 - \partial_z U^0 (\Delta r)^0 + \partial_{sz} U^0 (\nabla r \cdot \nabla s)^0 + D_u f(U^0) U^1,$$

where $(\Delta r)^0(t, z, s) = \Delta r(t, x_\varepsilon; \varepsilon)_{|\varepsilon=0}$ and $(\nabla r \cdot \nabla s)^0(t, z, s) = (\nabla r \cdot \nabla s)(t, x_\varepsilon; \varepsilon)_{|\varepsilon=0}$. Due to the orthogonality of the parametrisation close to the interface $\Gamma(t; 0)$, $(\nabla r \cdot \nabla s)^0$ vanishes, and we find

$$W^0 + \partial_z U^0 (\nabla r)^0 = -\partial_{zz} U^1 + D_u f(U^0) \cdot U^1.$$

We observe that $\partial_z U^0$ satisfies the corresponding homogeneous equation, and thus the solvability condition implies that $\int_{-\infty}^{+\infty} (W^0 + \partial_z U^0 (\Delta r)^0) \cdot \partial_z U^0 \, dz = 0$. Since W^0 and $(\Delta r)^0$ are independent of z, this gives the scalar relation

$$W^0 \cdot U^0 |_{z=-\infty}^{z=+\infty} + (\Delta r)^0 \int_{-\infty}^{+\infty} |\partial_z U^0|^2 \, dz = 0.$$

We observe that $(\Delta r)^0 = \kappa$, where κ is n times the mean curvature of $\Gamma(t; 0)$ and $\int_{-\infty}^{+\infty} |\partial_z U^0|^2\, dz$ was defined to be σ, earlier in this section. If ψ is the prototype potential, this solvability relation may be rewritten as $(W_k^0 - W_j^0)(1 - N\delta) = -\kappa\sigma$.

Matching inner and outer expansions

We want to match an inner expansion u_{in} around $\Gamma(t) = \Gamma(t; 0)$ with spatial normal $v(t, \cdot)$ to two corresponding outer expansions $u_{\text{out},\pm}$. Thus we assume $\Gamma(t; 0)$ locally to separate two bulk regions $\Omega_\pm(t)$. In $\Omega_\pm$ we assume that u takes the form $u_{\text{out},\pm}$ as given by the outer expansion, and we assume that all terms of the expansion together with their derivatives have limits as Γ is approached. Around Γ, we assume that u takes the form u_{in} as in the inner expansion.

Now, let z and $s \in \Gamma(t; 0)$ be given and set x_ε be the corresponding image of (z, s) under the reparametrisation.

In the matching region, we have two representations of the solution, namely

$$u_{\text{out},\pm}(t, x_\varepsilon; \varepsilon) = \sum_{J=0}^{\infty} \frac{1}{J!} \frac{d}{(d\varepsilon)^J} u_{\text{out},\pm}(t, x_\varepsilon; \varepsilon)_{|\varepsilon=0} \varepsilon^J =: \sum_{J=0}^{\infty} P_\pm^J(t, z, s)\varepsilon^J,$$

$$w_{\text{out},\pm}(t, x_\varepsilon; \varepsilon) = \sum_{J=0}^{\infty} \frac{1}{J!} \frac{d}{(d\varepsilon)^J} w_{\text{out},\pm}(t, x_\varepsilon; \varepsilon)_{|\varepsilon=0} \varepsilon^J =: \sum_{J=0}^{\infty} Q_\pm^J(t, z, s)\varepsilon^J,$$

and

$$u_{\text{in}}(t, x_\varepsilon; \varepsilon) = \sum_{J=0}^{\infty} \varepsilon^J U^J(t, z, s),$$

$$w_{\text{in}}(t, x_\varepsilon; \varepsilon) = \sum_{J=0}^{\infty} \varepsilon^J W^J(t, z, s).$$

We note that $P_\pm^K$ and $Q_\pm^K$ are polynomials of degree K in z.

The first matching conditions are

$$\lim_{z \to \pm\infty} (U^0(t, z, s) - u^0_{\text{out},\pm}(t, s)) = 0, \quad \lim_{z \to \pm\infty} (W^0(t, z, s) - w^0_{\text{out},\pm}(t, s)) = 0.$$

Since U^0 only depends on z, and $u^0_{\text{out},\pm}$ are constant, and since W^0 is independent of z, this implies

$$\lim_{z \to \pm\infty} U^0(z) = u^0_{\text{out},\pm},$$

$$W^0(t, s) = w^0_{\text{out},\pm}(t, s).$$

Thus, if the stationary wave solution U^0 connects $A(j)$ to $A(k)$, then $u^0_{\text{out},-} = A(j)$ and $u^0_{\text{out},+} = A(k)$. In addition $w^0_{\text{out},+} = w^0_{\text{out},-}$, which implies continuity of the limits of the chemical potentials across Γ.

The second matching conditions are

$$\lim_{z \to \pm\infty} (U^1(t, z, s) - (u^1_{\text{out},\pm}(t, s) + \nabla_x u^0_{\text{out},\pm}(t, s) \cdot v(t, s)z)) = 0,$$

$$\lim_{z \to \pm\infty} (W^1(t, z, s) - (w^1_{\text{out},\pm}(t, s) + \nabla_x w^0_{\text{out},\pm}(t, s) \cdot v(t, s)z)) = 0,$$

with $\nabla_x w^0_{\text{out},\pm}(t, s)$ being the Euclidean gradient of $w^0_{\text{out},\pm}(t, \cdot)$ evaluated at the point

$s \in \Gamma(t)$. In particular, this implies that

$$\lim_{z \to \pm\infty} \partial_z W^1(t, z, s) = \nabla_x w^0_{\text{out},\pm}(t, s) \cdot \nu(t, s),$$

and thus

$$\partial_z W^1|_{z=-\infty}^{z=+\infty} = \nabla_x(w^0_{\text{out},+} - w^0_{\text{out},-}) \cdot \nu.$$

Triple junction expansion and matching to the transition layer solutions in $\mathbf{R}^2$

We now assume that the space dimension n is equal to 2. We construct a solution in the neighbourhood of a triple point, where three phases meet, each phase corresponding to one of the three different values $A(j)$, $A(k)$ and $A(l)$. We follow the idea of [**30**] and [**5**]. Assume that $\Gamma_{jk}(t; \varepsilon)$, $\Gamma_{kl}(t; \varepsilon)$ and $\Gamma_{lj}(t; \varepsilon)$ are three smoothly evolving curves that meet exactly at one point $m_{jkl}(t; \varepsilon)$. We use (a, b) for any of the three pairs (j, k), (k, l) and (l, j). On each $\Gamma_{ab}(t; \varepsilon)$, we choose the normal $\nu_{ab}(t, \cdot; \varepsilon)$ to point into the $A(b)$-phase. We introduce the moving rescaled coordinates $y(t, x; \varepsilon) := (x - m_{jkl}(t; \varepsilon))/\varepsilon$.

We make the ansatz

$$u_{tp}(t, x; \varepsilon) = \sum_{K=0}^{\infty} \varepsilon^K \mathscr{U}^K(t, y(t, x; \varepsilon)), \quad w_{tp}(t, x; \varepsilon) = \sum_{K=0}^{\infty} \varepsilon^K \mathscr{W}^K(t, y(t, x; \varepsilon)),$$

with $\mathscr{U}^0 \in \Sigma$ and $\mathscr{U}^{K \geqq 1}, \mathscr{W}^{K \geqq 0} \in T\Sigma$. We substitute this into the Cahn–Hilliard system and then expand y in powers of ε.

The $\mathscr{O}(1/\varepsilon^2)$-equation of (I) reads

$$0 = \Delta_y \mathscr{W}^0.$$

We want $\mathscr{W}^0$ to be bounded, and thus Liouville's Theorem gives $\mathscr{W}^0(t, y) = \mathscr{W}^0(t)$.

The $\mathscr{O}(1/\varepsilon)$-equation of (II) reads

$$0 = -\Delta_y \mathscr{U}^0 + f(\mathscr{U}^0).$$

We are looking for a solution of this equation that connects $A(j)$ to $A(k)$ at $+\infty$ across Γ_{jk}, $A(k)$ to $A(l)$ at $+\infty$ of Γ_{kl} and $A(l)$ to $A(j)$ at $+\infty$ of Γ_{lj} in the form of the associated one-dimensional stationary wave solutions. Such a solution exists only if the angle condition

$$\sigma_{jk} \nu^0_{jk} + \sigma_{kl} \nu^0_{kl} + \sigma_{lj} \nu^0_{lj} = 0$$

is satisfied. This identity admits a solution if and only if the coefficients σ_{ab} fulfil $\sigma_{ab} + \sigma_{bc} \geqq \sigma_{ca}$ for any cyclic permutation (a, b, c) of (j, k, l). But since, in the present case, σ_{ab} can be characterised as $d(A(j), A(k))$ for some suitably chosen weighted distance d involving the potential ψ (see [**3**]), here this constraint is always met. In the case of the prototype potential ψ (and in fact any symmetric potential) the angle condition takes the equivalent form

$$\nu^0_{jk} \cdot \nu^0_{kl} = \nu^0_{kl} \cdot \nu^0_{lj} = \nu^0_{lj} \cdot \nu^0_{jk} = -\frac{1}{2} = \cos 120°.$$

Existence of solutions $\mathscr{U}^0$ provided the angle condition is met is shown in [**4**] for the case of a symmetric potential and two spatial dimensions. For a formal argument showing the necessity of the angle condition, we refer to [**5**]. For completeness, we include the derivation of the necessity in an Appendix. Since the angle condition is necessary for the first-order term to exist, it plays the role of a compatibility condition.

Triple junction lines and quadruple junction points in $\mathbf{R}^3$

The formal asymptotic expansion near a triple line can be carried out following the same ideas as near a triple point in $\mathbf{R}^2$. We only have to introduce an additional coordinate s in the direction of a triple line where three bulk phases together with three surfaces meet. We find that in the plane perpendicular to the triple line the above two-dimensional analysis holds true and that the lowest order term in the expansion does not depend on s. As a result, we find the angle condition in the plane perpendicular to the triple line.

At a quadruple point in $\mathbf{R}^3$ where four bulk phases meet, a similar expansion as at a triple point in $\mathbf{R}^2$ can be performed. But the whole geometry near a quadruple point is already determined by the geometry around triple lines (cf. the second part of Section 3) and thus no new information is contained in this expansion.

Boundary layer expansion

The boundary layer expansion can be performed as in [**30**], resulting in a Neumann condition for w and a right-angle condition for the intersection of the surface with the fixed boundary.

3. The limit problem and its geometric properties

The formal asymptotic expansion suggests a geometrical model as the limit of the Cahn–Hilliard system. This problem consists in determining a decomposition of Ω into bulk regions (see (A) below), surfaces (see (B) below), triple junction lines (see (C) below) and quadruple junctions (see (D) below). In the case of triple junction lines, we will rule out that any of the three bulk phases meeting there locally splits into several connected components, i.e. as we turn around a triple junction line, we only want to meet any of the three phases once. Mathematically, this will be captured by asking that in a neighbourhood of a triple junction line any of the three phases occupies a connected set. The same applies for quadruple junctions. We write down this geometrical problem in $\mathbf{R}^3$ when there are no singularities such as merging quadruple junctions or self-intersections of interfaces. The evolution of the interfaces between the bulk regions is determined through a system of partial differential equations for the chemical potential w. This system can be thought of as a generalisation of the two-phase Mullins–Sekerka problem.

Definition 3.1 (multi-phase Mullins–Sekerka system). Let $\{\sigma_{ab} \in \mathbf{R} : 1 \leqq a < b \leqq N\}$ be given, constant surface tensions. We say that $(\Omega_b)_{b=1,\dots,N} \subset [0, T] \times \Omega$ with $\Omega_b := \bigcup_{s \in (0,T)} \{s\} \times \Omega_b(s)$ is *a strong solution of the multi-phase Mullins–Sekerka problem*, if

- for every $t \in [0, T]$ the sets $\Omega_b(t) \subset \Omega$ are open, disjoint and $\Omega = \bigcup_{b=1}^{N} \overline{\Omega_b}(t)$;
- for all $x \in \Omega$ and for all $t \in (0, T)$ there exists $r > 0$ and $\tau > 0$ such that for $U := (t - \tau, t + \tau) \times B_r(x)$:

  (A) $U \subset \overline{\Omega_a}$ for some a;

or

(B) $U\cap\overline{\Omega_a}\cap\overline{\Omega_b}$ is a smoothly evolving surface $s\mapsto\Gamma_{ab}^{(x,t)}(s)$ for two distinct a and b and $U\cap\overline{\Omega_c}=\varnothing$ for all $c\neq a, b$;

or

(C) $U\cap\overline{\Omega_a}\cap\overline{\Omega_b}\cap\overline{\Omega_c}$ is a smoothly evolving triple junction curve $s\mapsto\Gamma_{abc}^{(x,t)}(s)$ for three distinct a, b and c, $U\cap\Omega_a$, $U\cap\Omega_b$, $U\cap\Omega_c$ are connected and $U\cap\overline{\Omega_d}=\varnothing$ for all $d\neq a, b, c$;

or

(D) $U\cap\overline{\Omega_a}\cap\overline{\Omega_b}\cap\overline{\Omega_c}\cap\overline{\Omega_d}$ is a smoothly evolving quadruple point $s\mapsto m_{abcd}(s)$ for four distinct a, b, c and d, $U\cap\Omega_a$, $U\cap\Omega_b$, $U\cap\Omega_c$, $U\cap\Omega_d$ are connected and $U\cap\overline{\Omega_e}=\varnothing$ for all $e\neq a, b, c, d$;

- for all $x\in\partial\Omega$ and all $t\in(0, T)$ there exists $r>0$ and $\tau>0$ such that (A), (B) or (C) above holds true with $B_r(x)$ substituted by $B_r(x)\cap\overline{\Omega}$ in the definition of U;
- for all $x\in\overline{\Omega}$ and $t=0$ the above holds true with $(t-\tau, t+\tau)$ substituted by $[0, \tau)$ in the definition of U;
- for every $t\in(0, T)$, there exists a continuous $w(t,\cdot):\Omega\to T\Sigma$ with $w(t,\cdot)\in C^2(\overline{\Omega_b(t)})$ for all b, such that

$$\Delta_x w=0 \quad \text{in } \Omega_b, \tag{3.1}$$

$$\partial_\nu w=0 \quad \text{on } (\partial\Omega)_T\cap\partial\Omega_b, \tag{3.2}$$

$$[\nabla_x w]_{|\Gamma_{ab}}\cdot\nu_{ab}=-v_{ab}[u]_{|\Gamma_{ab}} \quad \text{on } \Gamma_{ab}, \tag{3.3}$$

$$w\cdot[u]_{|\Gamma_{ab}}=-\sigma_{ab}\kappa_{ab} \quad \text{on } \Gamma_{ab}, \tag{3.4}$$

$$\nu_{ab}\cdot\nu=0 \quad \text{on } (\partial\Omega)_T\cap\partial\Gamma_{ab}, \tag{3.5}$$

$$\sigma_{ab}\nu_{ab}+\sigma_{bc}\nu_{bc}+\sigma_{ca}\nu_{ca}=0 \quad \text{on } \Gamma_{abc}, \tag{3.6}$$

with $u=A(b)$ in Ω_b and where $\Gamma_{ab}=\bigcup_{(x,t):\text{property}\,B}\Gamma_{ab}^{(x,t}$ and $\Gamma_{abc}=\bigcup_{(x,t):\text{property}\,C}\Gamma_{abc}^{(x,t)}$.

In the above, ν_{ab} is the spatial normal, κ_{ab} is the sum of principal curvatures and v_{ab} is the normal velocity of the surface Γ_{ab}, the normal pointing from Ω_a to Ω_b and $[f]_{|\Gamma}(x):=\lim_{h\to 0}f(x+h\nu)-f(x-h\nu)$ denotes the jump in normal direction across a surface Γ.

Remarks 3.2. (i) The relations (3.1), (3.2) and (3.3) are the continuity equation for the chemical potential w, relation (3.3) being a kinetic condition of Stefan-type and relations (3.4), (3.5) and (3.6) are the Gibbs–Thomson law. Relation (3.6) is also known as Young's law.

(ii) We point out that the proposed model is consistent with the asymptotic analysis. Relation (3.1) results from the first term of the outer expansion of w, relations (3.2) and (3.5) from the boundary layer expansion of w and u, relation (3.3) from the second-order term of the transition layer expansion of w together with the matching condition, relation (3.4) from the second-order term of the inner expansion of u and the compatibility condition and relation (3.6) from the compatibility condition of the first-order term in the triple point expansion of u.

(iii) In the case of the prototype potential, only the a-th and b-th component of $[u]_{|\Gamma_{ab}}$ are nonzero, and we may rewrite the interfacial conditions (3.3) and (3.4) componentwise as

$$[\nabla_x w_i]_{|\Gamma_{ab}} \cdot \nu_{ab} = 0 \quad \text{for } i \neq a, b,$$

$$[\nabla_x w_b]_{|\Gamma_{ab}} \cdot \nu_{ab} = -v_{ab}(1 - N\delta) \quad \text{and} \quad [\nabla_x w_a]_{|\Gamma_{ab}} \cdot \nu_{ab} = v_{ab}(1 - N\delta),$$

$$(w_b - w_a)_{|\Gamma_{ab}}(1 - N\delta) = -\sigma\kappa_{ab}.$$

Consequently, we find $[\nabla_x(w_b + w_a)]_{|\Gamma_{ab}} \cdot \nu_{ab} = 0$ and $[\nabla_x(w_b - w_a)]_{|\Gamma_{ab}} \cdot \nu_{ab} = -2v_{ab}(1 - N\delta)$. Thus the i-th component of w ($i \neq a, b$) and the sum of the a-th and b-th component of w satisfy $\Delta w_i = 0$ and $\Delta(w_a + w_b) = 0$ in $\Omega_a \cup \Omega_b$, respectively, whereas the difference of the a-th and b-th component of w is proportional to the curvature on Γ_{ab}, and the jump of its normal derivative on Γ_{ab} gives the velocity.

(iv) If the potential is symmetric, then $\sigma = \sigma_{ab}$ on each interface, and the triple junction condition (3.6) transforms into the condition that the surfaces meet at a 120° angle.

(v) An analogous definition can be made for a two-dimensional problem with bulk phases separated by interfacial lines which may meet in triple junction points.

Quadruple junction points in $\mathbf{R}^3$

The whole geometry and dynamics near a quadruple point is already given by the geometry of the higher-dimensional objects. In particular, the angles between the triple lines meeting at a quadruple point are determined by the angle conditions which hold on the triple lines themselves. For simplicity, we will only explain the symmetric case here. At a quadruple point $m(t)$ necessarily four triples lines Γ_{jkl}, Γ_{kml}, Γ_{lmj} and Γ_{jmk} meet. For any of these triples (a, b, c), we denote the tangent to Γ_{abc} by τ_{abc}, pointing away from the quadruple point. At any triple line Γ_{abc} three surfaces Γ_{ab}, Γ_{bc} and Γ_{ca} meet. The angle condition at the triple line Γ_{abc} implies $\nu_{ab} \cdot \nu_{bc} = \nu_{bc} \cdot \nu_{ca} = \nu_{ca} \cdot \nu_{ab} = -\frac{1}{2}$. Using the three normals at each triple line, we can represent the tangents to the triple lines:

$$\tau_{jkl} = \frac{\nu_{jk} \times \nu_{kl}}{|\nu_{jk} \times \nu_{kl}|} = \frac{2}{\sqrt{3}} \nu_{jk} + \nu_{kl},$$

$$\tau_{kml} = \frac{\nu_{km} + \nu_{ml}}{|\nu_{km} \times \nu_{ml}|} = \frac{2}{\sqrt{3}} \nu_{km} \times \nu_{ml},$$

$$\tau_{lmj} = \frac{\nu_{lm} \times \nu_{mj}}{|\nu_{lm} \times \nu_{mj}|} = \frac{2}{\sqrt{3}} \nu_{lm} \times \nu_{mj},$$

$$\tau_{jmk} = \frac{\nu_{jm} \times \nu_{mk}}{|\nu_{jm} \times \nu_{mk}|} = \frac{2}{\sqrt{3}} \nu_{jm} \times \nu_{mk}.$$

We recall that $\nu_{ab} = -\nu_{ba}$ is the normal to any surface Γ_{ab} pointing from phase a to phase b. The well-known formula $(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$ for scalar and vector products implies

$$\tau_{jkl} \cdot \tau_{kml} = \frac{4}{3}((\nu_{jk} \cdot \nu_{km})(\nu_{kl} \cdot \nu_{ml}) - (\nu_{jk} \cdot \nu_{ml})(\nu_{kl} \cdot \nu_{km})).$$

Using $\nu_{kl} \cdot \nu_{ml} = -\nu_{kl} \cdot \nu_{lm} = \frac{1}{2}$ and $\nu_{jk} \cdot \nu_{ml} = \nu_{jk} \cdot (-\nu_{lk} - \nu_{km}) = -\frac{1}{2} + \frac{1}{2} = 0$, we finally obtain

$$\tau_{jkl} \cdot \tau_{kml} = -\frac{1}{3}.$$

In a similar way, it is possible to compute that the scalar products of all other combinations of tangent vectors at the quadruple point is equal to $-\frac{1}{3}$. This implies that the angle between the triple lines at a quadruple point is such that its cosine equals $-\frac{1}{3}$. This angle is well known in the theory of minimal surfaces. A system of surfaces which locally minimises the area consists of minimal surfaces which meet at triple lines where they satisfy a 120° angle condition, and these triple lines again meet in quadruple points with an angle whose cosine is $-\frac{1}{3}$. We refer to **[24]** for a nice description.

Finally, we just state the law that holds at a quadruple junction when the surface tensions are not identical:

$$\delta_{jkl}\tau_{jkl} + \delta_{kml}\tau_{kml} + \delta_{lmj}\tau_{lmj} + \delta_{jmk}\tau_{jmk} = 0,$$

where δ might be thought of as the surface tensions supported by a line singularity and which relate to the interfacial surface tensions via the formula

$$\delta_{jkl} = \left(2 \sum_{abc = jkl, klj, ljk} \sigma_{ab}^2 \sigma_{bc}^2 - \sum_{ab = jk, kl, lj} \sigma_{ab}^4\right)^{\frac{1}{2}}.$$

The term in brackets is non-negative if the surface tensions fulfil the subadditivity condition $\sigma_{ab} + \sigma_{bc} \geqq \sigma_{ca}$.

Geometric properties

For simplicity of presentation, we only discuss the case of the prototype potential, i.e. we assume symmetry and the special form of the stable phases $A(j)$. We note the following geometric properties at the triple lines or points, which formally follow from the definition.

(i) Due to (3.4) we may add the three identities $w_b - w_a = -\kappa_{ab}\sigma/(1-\delta N)$ at any point m on a triple line and obtain

$$\kappa_{jk} + \kappa_{kl} + \kappa_{lj} = 0 \quad \text{at } m.$$

(i) We may determine the velocity of triple lines and points. Let Γ_{jkl} be a triple line with tangent τ_{jkl}. We parametrise $\Gamma_{jkl}(t)$ by $m(t,\cdot)$ such that $\dot{m} \cdot \tau_{jkl} = 0$. At the triple line Γ_{jkl}, three interfaces Γ_{jk}, Γ_{kl} and Γ_{lj} meet, keeping the 120°-angle condition. Since m stays on any of the interfaces Γ_{ab}, we obtain

$$\dot{m} \cdot \nu_{ab} = v_{ab} \quad \text{for } (a,b) = (j,k), (k,l), (l,j).$$

Since $\nu_{ab} \cdot \nu_{bc} = -\frac{1}{2}$, we have $\dot{m} = (\frac{4}{3})(v_{ab} + \frac{1}{2}v_{bc})\nu_{ab} + (\frac{4}{3})(v_{bc} + \frac{1}{2}v_{ab})\nu_{bc}$. We add these identities over any cyclic permutation of (j,k,l) and obtain $3\dot{m} = (\frac{4}{3})\Sigma_{(a,b,c)} v_{ab}(2\nu_{ab} + (\frac{1}{2})\nu_{bc} + (\frac{1}{2})\nu_{ca})$. Since $\nu_{ab} = -\nu_{bc} - \nu_{ca}$, this simplifies into $\dot{m} = (\frac{2}{3})\Sigma_{(a,b)} v_{ab}\nu_{ab}$. Using the formula (3.3) for v_{ab} gives the evolution law for the triple line

$$3\dot{m} = -\frac{1}{(1-N\delta)} \sum_{(a,b)} ([\nabla_x(w_b - w_a)]_{\Gamma_{ab}} \cdot \nu_{ab})\nu_{ab},$$

where summation is over $(a,b)=(j,k),(k,l),(l,j)$. The same formula is true for the evolution of a triple point in $\mathbf{R}^2$.

(iii) In addition, we may calculate some nonlocal quantities in $\mathbf{R}^2$. Obviously the total area of the individual phases is conserved. Furthermore, we find that the total length of the interfacial curve decreases: due to the angle condition (3.5) the formula for the change of total length is

$$\frac{d}{dt}L(t):=\frac{d}{dt}\sum_{(a,b)}\mathscr{H}^1(\Gamma_{ab})=\sum_{(a,b)}\int_{\Gamma_{ab}}\kappa_{ab}v_{ab}\,do+\sum_{(a,b)}\dot{m}_{jkl}\cdot\tau_{ab}.$$

We substitute the relations (3.3) and (3.4) into this formula and obtain

$$\frac{d}{dt}L(t)=\frac{1}{\sigma}\sum_{(a,b)}\int_{\Gamma_{ab}}w\cdot[\partial_{\nu_{ab}}w]_{|\Gamma_{ab}}\,do+\dot{m}_{jkl}\cdot\sum_{(a,b)}\tau_{ab}.$$

Due to the angle condition (3.6), the last term vanishes. We note that $\Gamma_{ab}\cup\Gamma_{bc}$ together with the appropriate portion of $\partial\Omega$ is the boundary of Ω_b. Thus, using the Neumann condition (3.2) we may rewrite this as

$$\frac{d}{dt}L(t)=-\frac{1}{\sigma}\sum_b\int_{\partial\Omega_b}(w\cdot\nabla w)\cdot n_b\,do,$$

where n_b is the exterior normal of Ω_b, and ∇w is the limit of ∇w from the interior of Ω_b. Thus the Divergence Theorem implies that

$$\frac{d}{dt}L(t)=-\frac{1}{\sigma}\sum_b\int_{\Omega_b}\operatorname{div}(w\cdot\nabla w)\,dx.$$

Using the bulk differential equation (3.1) finally gives

$$\frac{d}{dt}L(t)=-\frac{1}{\sigma}\int_{\Omega}|\nabla w|^2\,dx.$$

For a nonsymmetric potential, we obtain that the weighted length $\Sigma_{(a,b)}\sigma_{ab}\mathscr{H}^1(\Gamma_{ab})$ decreases and in $\mathbf{R}^3$ the result is that $\Sigma_{(a,b)}\sigma_{ab}\mathscr{H}^2(\Gamma_{ab})$ decreases.

4. Weak formulation for the multi-phase Mullins–Sekerka system

In order to give a weak formulation of the problem, we first give a notion of an admissible partition of Ω. To any partition of Ω into N phases, there is associated a vector of characteristic functions $\chi=(\chi_b)_{b=1,\ldots,N}$, so that χ_b is the characteristic function of phase b. Vice versa, a vector of characteristic functions, with the property that for any spatial point exactly one of the χ_b's is 1, gives a decomposition of Ω into N phases. We will ask this vector to be of bounded variation, or equivalently, ask any of the sets that make up the decomposition of Ω to be of finite perimeter. This concept implies the following definition of admissible partitions:

$$K:=\left\{\chi=(\chi_b)_{b=1,\ldots,N}\in(BV(\Omega))^N\;\middle|\;\sum_b\chi_b=1,\text{ and }\chi_b(1-\chi_b)=0\text{ for all }b\right\}.$$

For any $\chi \in K$, we may define the system of generalised normal vectors

$$\nu_b := \frac{\nabla \chi_b}{|\nabla \chi_b|} \in L^1(|\nabla \chi_b|) \quad \text{for } b = 1, \ldots, N.$$

We assume that for $a, b = 1, \ldots, N$ we are given non-negative constants σ_{ab}, with $\sigma_{aa} = 0$ and $\sigma_{ab} = \sigma_{ba}$, which satisfy

$$\sigma_{ac} \leqq \sigma_{ab} + \sigma_{bc}$$

for all $a, b, c = 1, \ldots, N$.

Definition 4.1. Assume that $\chi^0 = (\chi_b^0)_{b=1,\ldots,N} \in K$ and set $u^0 := \Sigma_{b=1}^N A(b)\chi_b^0$.
We say that

$$\chi = (\chi_b)_{b=1,\ldots,N} \in L^\infty(0, T; K)$$

is *a weak solution of the multi-phase Mullins–Sekerka system for the initial data* χ^0, if there exists an associated vector of weighted chemical potential differences

$$w = (w_b)_{b=1,\ldots,N} \in L^2(0, T; H^1(\Omega)^N),$$

such that for $u := \Sigma_{b=1}^N A(b)\chi_b$

$$\int_0^T \int_\Omega -u \cdot \partial_t \xi \, dx\, dt - \int_\Omega u^0 \xi(0, \cdot)\, dx + \int_0^T \int_\Omega \nabla w \cdot \nabla \xi \, dx\, dt = 0 \tag{4.1}$$

for all $\xi = (\xi_b)_{b=1,\ldots,N} \in (C_0^\infty([0, T) \times \overline{\Omega}))^N$, and

$$\int_0^T \sum_{a,b=1, a<b}^N \sigma_{ab} \int_\Omega (\operatorname{div} \zeta - \nu \cdot \nabla\zeta \nu)\mu_{ab} = \int_0^T \sum_{b=1}^N \int_\Omega \nabla(w \cdot u) \cdot \zeta \chi_b \, dx\, dt \tag{4.2}$$

for all $\zeta \in (C^\infty([0, T) \times \overline{\Omega}))^n$ with $\operatorname{div} \zeta = 0$ in Ω and $\xi \cdot \nu = 0$ on $\partial\Omega$. Here

$$\mu_{ab} := \frac{1}{2}(|\nabla \chi_a| + |\nabla \chi_b| - |\nabla(\chi_a + \chi_b)|)$$

and $\nu := \nabla \chi_a / |\nabla \chi_a|$ on the support of μ_{ab}.

Remarks 4.2. (i) A weak formulation of the Gibbs–Thomson law (4.2) was introduced by Luckhaus [26].

(ii) Following Vol'pert [37], we have $\nabla\chi_a/|\nabla\chi_a| + \nabla\chi_b/|\nabla\chi_b| = 0$ almost everywhere with respect to the measure μ_{ab}.

(iii) In addition, we point out that the chemical potential at each time is only unique up to the addition of a constant.

The next proposition shows in the case of three phases with a single triple line that a smooth weak solution satisfies the strong formulation of the multi-phase Mullins–Sekerka problem. This proof easily extends to the situation of more phases with finitely many triple lines that may or may not meet in quadruple junctions.

Proposition 4.3. *Assume that the space dimension is* 3 *and that* $\chi \in L^\infty(0, T; K)$ *is a weak solution of the multi-phase Mullins–Sekerka system. Assume that the following structure assumptions hold true*: $\chi_b \equiv 0$ *for* $b \notin \{j, k, l\}$ *and* $\chi_b \not\equiv 0$ *for* $b = j$, k *and* l. *For* $b = j$, k *and* l, *set* $\Omega_b(t) := \{\chi_b(t, \cdot) = 1\}$ *and* $\Gamma_b(t) := \partial\Omega_b(t) \cap \Omega$, *and for* $(a, b) = (j, k)$, (k, l) *and* (l, j), *set* $\Gamma_{ab} := \Gamma_b \cap \Gamma_a$.

We assume that $\Gamma_{ab}(t)$ are smoothly evolving curves, that meet at exactly one smoothly evolving triple junction curve $\Gamma_{jkl}(t)$ and that intersect the fixed boundary in a smoothly evolving curve. We choose the (spatial) normal ν_{ab} on Γ_{ab} to point into Ω_b. We define the curvature $\kappa_{ab} := \operatorname{div}_{\Gamma_{ab}} \nu_{ab}$ and the normal velocity v_{ab} on Γ_{ab}. In addition, we assume that for all $b = j$, k and l the associated vector of chemical potentials satisfies

$$w(t, \cdot) \in (C^2(\overline{\Omega_b(t)}) \cap C^0(\overline{\Omega}))^N \quad \textit{for all } t \in [0, T].$$

Then for any t there exists a constant vector $c(t) \in \mathbf{R}^N$ such that $\chi = (\chi_b)_{b=1,\dots,N}$ and $w - c = (w_b - c_b)_{b=1,\dots,N}$ is a strong solution of the multi-phase Mullins–Sekerka system.

Proof. If not otherwise specified, we use b for j, k and l, (a, b) for (j, k), (k, l) and (l, j) and (a, b, c) for (j, k, l) and its cyclic permutations.

Due to the structure and smoothness assumptions of this proposition, $\partial_t u = \Sigma_b A(b) \partial_t \chi_b$ and $\partial_t \chi_b = -v_b \, d\mathscr{H}^2 \llcorner \Gamma_b$, where v_b is the velocity of Γ_b in the direction of ν_b. Since Γ_b may be decomposed into Γ_{ab} and Γ_{bc}, we obtain that

$$\partial_t u = -\sum_{(a,b)} [u]_{|\Gamma_{ab}} v_{ab} \, d\mathscr{H}^2 \llcorner \Gamma_{ab}.$$

In addition,

$$\Delta w = \sum_b \Delta w_{|\Omega_b} + \sum_{(a,b)} [\nabla w]_{|\Gamma_{ab}} \cdot \nu_{ab} \, d\mathscr{H}^2 \llcorner \Gamma_{ab} - \partial_\nu w \, d\mathscr{H}^2 \llcorner \partial\Omega.$$

Using this, we deduce from (4.1) the equation

$$\partial_t u - \Delta w = 0.$$

This thus implies

$$\Delta w = 0 \quad \text{in } \Omega_b,$$

$$\partial_\nu w = 0 \quad \text{on } \partial\Omega,$$

$$[\nabla w]_{|\Gamma_{ab}} \cdot \nu_{ab} = -[u]_{\Gamma_{ab}} v_{ab} \quad \text{on } \Gamma_{ab}.$$

The first equation is relation (3.1) of the strong formulation of the multi-phase Mullins–Sekerka system, the second equation is relation (3.2), and the third equation is relation (3.3).

Now we proceed with (4.2). Under the smoothness assumptions of this proposition, $\mu_{ab} = d\mathscr{H}^2 \llcorner \Gamma_{ab}$ and $(\operatorname{div} \zeta - \nu \cdot \nabla\zeta\nu) = \operatorname{div}_{\Gamma_{ab}} \zeta$ on Γ_{ab}. Thus we obtain

$$\begin{aligned} E_{ab}(\zeta) &:= \int_\Omega (\operatorname{div} \zeta - \nu \cdot \nabla\zeta\nu) \mu_{ab} = \int_{\Gamma_{ab}} \operatorname{div}_{\Gamma_{ab}} \zeta \, d\mathscr{H}^2 \\ &= \int_{\Gamma_{ab}} \operatorname{div}_{\Gamma_{ab}} \nu_{ab} (\zeta \cdot \nu_{ab}) \, d\mathscr{H}^2 + \int_{\partial\Gamma_{ab}} \zeta \cdot \tau_{ab} \, d\mathscr{H}^1. \end{aligned}$$

In the second identity, we used the formula

$$\int_\Gamma \operatorname{div}_\Gamma \zeta \, d\mathscr{H}^2 = \int_\Gamma \operatorname{div}_\Gamma n (\zeta \cdot n) \, d\mathscr{H}^2 + \int_{\partial\Gamma} \zeta \cdot \tau \, d\mathscr{H}^1,$$

where τ is the normal to $\partial\Gamma$ lying in the tangent plane of Γ.

Now we assume that the test function ζ satisfies $\zeta \cdot \nu = 0$ on $\partial\Omega$. Then we get

$$
\begin{aligned}
L_b(\zeta) &:= \int_\Omega \chi_b \operatorname{div}((w \cdot u)\zeta)\, dx \\
&= -\int_\Omega (w \cdot u)\zeta \cdot \nabla \chi_b = -\int_{\Gamma_b} (w \cdot u)\zeta \cdot n_b \, d\mathscr{H}^2 \\
&= -\int_{\Gamma_{ab}} (w \cdot u)^+ \zeta \cdot \nu_{ab}\, d\mathscr{H}^2 + \int_{\Gamma_{bc}} (w \cdot u)^- \zeta \cdot \nu_{bc}\, d\mathscr{H}^2.
\end{aligned}
$$

Putting everything together, we obtain the formula

$$
\begin{aligned}
\sum_{(a,b)} \sigma_{ab} E_{ab}(\zeta) - \sum_{b=1}^{N} L_b(\zeta) &= \sum_{(a,b)} \int_{\Gamma_{ab}} (\sigma_{ab}\kappa_{ab} + w \cdot [u]_{|\Gamma_{ab}})(\zeta \cdot \nu_{ab})\, d\mathscr{H}^2 \\
&\quad + \sum_{(a,b)} \sigma_{ab} \int_{\partial\Gamma_{ab} \cap \partial\Omega} \zeta \cdot \tau_{ab}\, d\mathscr{H}^1 + \int_{\Gamma_{jkl}} \zeta \cdot \sum_{(a,b)} \sigma_{ab}\tau_{ab}\, d\mathscr{H}^1.
\end{aligned}
$$

We conclude that for any (a, b) and all t there exists some constant $c_{ab}(t)$ such that for all t

$$
\begin{aligned}
-\sigma_{ab}\kappa_{ab} &= w \cdot [u]_{|\Gamma_{ab}} + c_{ab} \quad \text{on } \Gamma_{ab}, \\
\tau_{ab} &= \nu \quad \text{on } \partial\Gamma_{ab} \cap \partial\Omega, \\
\sum_{(a,b)} \sigma_{ab}\tau_{ab} &= 0 \quad \text{on } \Gamma_{jkl}.
\end{aligned}
$$

We note that the $c_{ab}(t)$ satisfy $c_{ij} + c_{jk} + c_{kl} = 0$ for all t. Thus the chemical potential can be modified by a constant so that all c_{ab} become 0.

The first equation is relation (3.4) of the strong formulation of the multi-phase Mullins–Sekerka system, the second equation implies the right-angle condition (3.5), and the last equation is the angle condition (3.6). □

5. An implicit time discretisation

From now on, for the clarity of the text, we will assume that $A(b) = e_b$ for $b = 1, \ldots, N$. This may always be achieved by a linear transformation. To construct approximate solutions, we apply an implicit time discretisation. This approach has been introduced by Luckhaus [26] for the Stefan problem and developed further for the mean curvature flow by Almgren, Taylor and Wang [2] and by Luckhaus and Sturzenhecker [27]. This method has also been used by Garcke and Sturzenhecker [22] for the multiphase evolution of a single equation and by Otto [29] for the dynamics of pattern formation in magnetic fluids. The main new features here are that we allow for multiple junctions and that we incorporate mass conservation by prescribing Neumann boundary data for the chemical potentials.

We proceed as follows: we choose a positive time step h. For $t < 0$, we set $\chi^h(t) := \chi^0$. Since the limit flow has the property that the volume of the individual phases is conserved, we introduce

$$K_0 := K \cap \left\{ \int_\Omega \chi \, dx = \int_\Omega \chi^0 \, dx \right\},$$

where the set K of admissible partitions of Ω was defined at the beginning of Section 4. Thus K_0 contains all admissible partitions of Ω with conservation of volume of all components relative to the initial data.

Now we assume that $\chi^h(t-h)$ were already defined. Our goal is to define $\chi^h(t)$ as the absolute minimiser of an appropriate functional.

DEFINITION 5.1. For $\chi \in K_0$, we define

$$\mathscr{E}^h(\chi, \chi^h(t-h)) := \int_\Omega \sum_{a,b=1,a<b}^{N} \sigma_{ab}\mu_{ab} + \frac{h}{2}\int_\Omega |\nabla w|^2 \, dx, \tag{5.1}$$

where $w = (w_a)_{a=1,\ldots,N} \in (H^1(\Omega))^N$ is the solution of

$$\begin{aligned} \chi_a - \chi_a^h(t-h) &= h\Delta w_a \quad \text{in } \Omega, \\ \frac{\partial w_a}{\partial \nu} &= 0 \quad \text{on } \partial\Omega, \\ \int_\Omega w_a \, dx &= 0. \end{aligned} \tag{5.2}$$

REMARK 5.2. We note that if $\chi = \chi^h(t)$, then (5.2) is the implicit time discretisation of $\partial_t \chi_a = \Delta w_a$.

In order to show the existence of an absolute minimiser of this functional in the class K_0, we will use the direct method. The following lemma gives the basic tools to show compactness and lower semicontinuity.

We shall need the measure theoretic supremum $\vee_{a=1}^N \mu_a$ (see [3]) of positive, regular measures μ_a, which for all open subsets $D \subset \Omega$ is given by

$$\left(\bigvee_{a=1}^N \mu_a \right)(D) := \sup \left\{ \sum_{a=1}^N \mu_a(B_a) \mid B_a \subset D, \text{ open, pairwise disjoint} \right\}.$$

Furthermore, we denote the reduced boundary of a set B (or better of an equivalence class of sets) by $\partial^* B$ (cf. [23]).

LEMMA 5.3. (i) *For all open sets $D \subset \Omega$, we have*

$$\int_D \mu_{ab} = \mathscr{H}^{n-1}(\partial^*\Omega_a \cap \partial^*\Omega_b \cap D),$$

where $\Omega_a := \{\chi_a = 1\}$, and

$$\int_D \sum_{a,b=1,a<b}^N \sigma_{ab}\mu_{ab} = \left(\bigvee_{a=1}^N \mu_a \right)(D),$$

where $\mu_a(D) = |\nabla(\sum_{b=1}^N \sigma_{ab}\chi_b)|(D)$.

(ii) *We have*

$$\mu_a(\Omega) \geqq \sigma_a |\nabla \chi_a|(\Omega),$$

where $\sigma_a := \min_{b=1,\ldots,N, b \neq a} \sigma_{ab}$.

Proof. (i) Vol'pert [37] and Baldo [3] prove that for all open sets $D \subset \Omega$,

$$\int_D |\nabla \chi_a| = \sum_{j=1, j\neq a}^{N} \mathscr{H}^{n-1}(\partial^*\Omega_a \cap \partial^*\Omega_j \cap D)$$

and

$$\int_D |\nabla(\chi_a + \chi_b)| = \sum_{j=1, i\neq a,b}^{N} \mathscr{H}^{n-1}(\partial^*\Omega_a \cap \partial^*\Omega_j \cap D) + \mathscr{H}^{n-1}(\partial^*\Omega_b \cap \partial^*\Omega_j \cap D).$$

This implies

$$\int_D \mu_{ab} = \mathscr{H}^{n-1}(\partial^*\Omega_a \cap \partial^*\Omega_b \cap D).$$

Using this identity, we apply [3, Proposition 2.2] to conclude.

(ii) An application of the coarea formula [23] gives

$$\begin{aligned}\mu_a(\Omega) &= \int_\Omega \left|\nabla\left(\sum_{b=1}^{N} \sigma_{ab}\chi_b\right)\right| = \int_{-\infty}^{\infty} P_\Omega\left(\left\{x \,\middle|\, \sum_{b=1}^{N} \sigma_{ab}\chi_b \leqq \lambda\right\}\right) d\lambda \\ &\geqq \int_0^{\sigma_a} P_\Omega\left(\left\{x \,\middle|\, \sum_{b=1}^{N} \sigma_{ab}\chi_b \leqq \lambda\right\}\right) d\lambda \\ &= \sigma_a P_\Omega(\Omega_a) = \sigma_a |\nabla \chi_a|(\Omega). \quad \square\end{aligned}$$

As a consequence of this lemma, we may show the existence of an absolute minimiser of $\mathscr{E}^h$.

LEMMA 5.4. *There exists* $\chi^h(t) \in K_0$ *such that*

$$\mathscr{E}^h(\chi^h(t), \chi^h(t-h)) = \inf_{\chi \in K_0} \mathscr{E}^h(\chi, \chi^h(t-h)).$$

Proof. Let $(\chi^m)_{m\in\mathbb{N}} \subset K_0$ be a minimising sequence of $\mathscr{E}^h(\chi, \chi^h(t-h))$. Since $(\vee_{a=1}^N \mu_a)(\Omega) \geqq \mu_b(\Omega)$ for any $1 \leqq b \leqq N$, we may use Lemma 5.3 to conclude that $(\chi^m)_m$ is uniformly bounded in $(BV(\Omega))^N$. Using the compactness of the embedding of $(BV(\Omega))^N$ into $(L^1(\Omega))^N$, we may select a subsequence (still denoted by χ^m) such that, for some $\chi \in K_0$,

$$\chi^m \to \chi \quad \text{in } (L^1(\Omega))^N \text{ and almost everywhere.}$$

Since all χ_a^m are characteristic functions, and therefore bounded, we may as well deduce that, for all $1 \leqq p < \infty$,

$$\chi^m \to \chi \quad \text{in } (L^p(\Omega))^N.$$

This implies for the corresponding w^m (cf. (5.2) of the definition of the functional $\mathscr{E}^h$)

$$\int_\Omega |\nabla w^m|^2\, dx \to \int_\Omega |\nabla w|^2\, dx.$$

It remains to show lower semicontinuity of the interfacial energy term. Since

$$\sum_{b=1}^{N} \sigma_{ab}\chi_b^m \to \sum_{b=1}^{N} \sigma_{ab}\chi_b \quad \text{in } L^1(\Omega),$$

we obtain for all open $D \subset \Omega$

$$\mu_a(D) = \int_D \left|\nabla\left(\sum_{b=1}^{N} \sigma_{ab}\chi_b\right)\right| \leqq \liminf_{m\to\infty} \int_D \left|\nabla\left(\sum_{b=1}^{N} \sigma_{ab}\chi_b^m\right)\right| = \liminf_{m\to\infty} \mu_a^m(D).$$

We now claim that

$$\left(\bigvee_{a=1}^{N} \mu_a\right)(\Omega) \leqq \liminf_{m\to\infty}\left(\bigvee_{a=1}^{N} \mu_a^m\right)(\Omega).$$

To see this, let B_a be open and pairwise disjoint sets with $\cup_{a=1}^N B_a \subset \Omega$. Then

$$\begin{aligned}\sum_{a=1}^{N} \mu_a(B_a) &\leqq \sum_{a=1}^{N} \liminf_{m\to\infty} \mu_a^m(B_a)\\ &\leqq \liminf_{m\to\infty} \sum_{a=1}^{N} \mu_a^m(B_a) \leqq \liminf_{m\to\infty}\left(\bigvee_{a=1}^{N} \mu_a^m\right)(\Omega).\end{aligned}$$

Taking the supremum over all choices of B_a implies the claim. The lower semicontinuity is now a direct consequence of the representation formula given in Lemma 5.3(i). □

Remark 5.5. Next, we derive the Euler–Lagrange equations for the absolute minimiser $\chi^h(t)$. Since we incorporated the volume constraint into the definition of the set K_0, we can only allow for variations which keep the volume of the individual phases unaltered. We therefore consider a family of deformations $\Phi(\tau, \cdot)$ of Ω, given by

$$\Phi(0, x) = x \quad \text{and} \quad \partial_\tau \Phi(\tau, x) = \zeta(\Phi(\tau, x))$$

for $x \in \Omega$ and $\tau \in [0, \tau_0]$. Here ζ is assumed to be an arbitrary smooth vector field $\zeta : \bar{\Omega} \to \mathbf{R}^n$ with $\operatorname{div} \zeta = 0$ in Ω and $\zeta \cdot \nu = 0$ on $\partial\Omega$. Since ζ is divergence free, deformations of Ω by $\Phi(\tau, \cdot)$ do not change the volume of the individual phases, and variations of $\mathscr{E}^h(\cdot, \chi^h(t-h))$ in directions given by Φ imply (cf. [**23**, **25**]).

$$\int_\Omega \sum_{a,b=1,a<b}^{N} \sigma_{ab}(\operatorname{div} \zeta - \nu^h \cdot \nabla\zeta\nu^h)\mu_{ab}^h = \sum_{a=1}^{N} \int_\Omega \nabla w_a^h \cdot \zeta\chi_a^h \, dx \tag{5.3}$$

for all divergence-free vector fields ζ.

Lemma 5.6 (*a-priori* estimates). (i) *For all $t = m \cdot h$ with $m \in N$,*

$$\sup_{0\leqq\tau\leqq t} \int_\Omega \sum_{a,b=1,a<b}^{N} \sigma_{ab}\mu_{ab}^h(\tau) + \frac{1}{2}\int_0^t \int_\Omega \sum_{a=1}^{N} |\nabla w_a^h(t)|^2 \, dx \, dt \leqq \int_\Omega \sum_{a,b=1,a<b}^{N} \sigma_{ab}\mu_{ab}^h(0).$$

(ii) *There exists some constant C only depending on σ and the initial data, such that*

$$\sup_{0\leqq\tau\leqq t} \int_\Omega \sum_{a=1}^{N} |\nabla\chi_a^h|(\tau) \leqq C.$$

Proof. By construction,

$$\mathscr{E}^h(\chi^h(ih), \chi^h((i-1)h)) \leqq \mathscr{E}^h(\chi^h((i-1)h), \chi^h((i-1)h).$$

Since χ^h and w^h are piecewise constant in time, summation over $i = 1, \ldots, m$ implies (i).

Assertion (ii) follows from Lemma 5.3. □

LEMMA 5.7. *There exists a subsequence $h \to 0$ and a limiting pair (χ, w) with $\chi \in L^\infty(0, T; K_0)$ and $w \in L^2(0, T; (H^1(\Omega))^N)$ such that*

$$\chi^h \to \chi \quad \text{in } (L^1((0, T) \times \Omega))^N \text{ and almost everywhere,}$$

$$\nabla\chi^h \stackrel{*}{\rightharpoonup} \nabla\chi \quad \text{weakly in the sense of Radon measures,}$$

$$w^h \rightharpoonup w \quad \text{weakly in } L^2(0, T; (H^1(\Omega))^N).$$

Proof. The second and third assertions follow directly from the *a-priori* bounds of Lemma 5.6 and the mean value condition $\int_\Omega w_a^h = 0$. To show the first assertion, we will use the Fréchet–Kolmogorov–Riesz ([**33**], [**15**; IV.8.Theorem 21]) compactness criterion. The uniform boundedness of χ^h in $(L^\infty(0, T; BV(\Omega)))^N$ guarantees control on spatial differences:

$$\int_0^T \int_{\Omega'} |\chi_a^h(t, x+y) - \chi_a^h(t, x)| \, dx \leqq |y| \int_0^T \int_\Omega |\nabla\chi_a^h| \leqq C|y|,$$

for all $y \in \mathbf{R}^n$ and $\Omega' \subset \Omega$, such that $\Omega' + y \subset \Omega$. Furthermore, we have for any $t = m \cdot h$ and $\tau = k \cdot h$

$$\begin{aligned} \|\chi_a^h(t) - \chi_a^h(t-\tau)\|_{H^1(\Omega)'} &= \left\| \int_\tau^t \Delta w_a^h(s) \, ds \right\|_{H^1(\Omega)'} \\ &\leqq \tau^{\frac{1}{2}} \left(\int_\tau^t \|\Delta w_a^h(s)\|_{H^1(\Omega)'}^2 \, ds \right)^{\frac{1}{2}} \\ &\leqq C\tau^{\frac{1}{2}} \|w_a^h\|_{L^2(0,t;H^1(\Omega))}. \end{aligned}$$

By interpolation, these two estimates together imply the compactness of χ^h in L^1 (see [**26**]). □

THEOREM 5.8. *Let $T > 0$ be arbitrary and assume that for $a = 1, \ldots, N$,*

$$\liminf_{h \to 0} \int_0^T \int_\Omega |\nabla\chi_a^h| \leqq \int_0^T \int_\Omega |\nabla\chi_a|.$$

Then the limit χ, constructed in Lemma 5.7, is a weak solution of the multiphase Mullins–Sekerka system with initial data χ^0.

REMARK 5.9. The assumption of this theorem implies that there is no loss of perimeter in the limit process. Similar conditions are necessary in the implicit time discretisation of the mean curvature flow (see [**2, 27**]). We also refer to [**22, 29**].

Proof of Theorem 5.8. We multiply the differential equation $\partial_t^{-h}\chi^h = \Delta w^h$ (cf. (5.2) in the definition of the functional $\mathscr{E}^h$) by a smooth test function $\xi = (\xi_a)_{a=1,\ldots,N}$ with compact support in $[0, T) \times \overline{\Omega}$, integrate in space and time and perform a discrete integration by parts. This gives

$$-\int_0^{T-h}\int_\Omega \chi^h(t)\frac{\xi(t+h)-\xi(t)}{h}\,dx\,dt-\frac{1}{h}\int_{-h}^0\chi^0\xi(t)\,dx\,dt$$
$$+\int_0^T\int_\Omega \nabla w^h\cdot\nabla\xi\,dx\,dt=0.$$

The strong convegence of χ^h and the weak convergence of w^h imply equation (4.1) of the definition of a weak solution.

The substantially more difficult part is to pass to the limit in the weak formulation of the curvature relation. The assumption of this theorem implies that in addition to the results of Lemma 5.7, for almost all $t\in(0,T)$,

$$\lim_{h\to 0}\int_\Omega|\nabla\chi_a^h|(t)\to\int_\Omega|\nabla\chi_a|(t),$$

i.e. for almost all t there is no loss of perimeter in the limit process.

We first show that this ensures the weak-* convergence of μ_{ab}^h to μ_{ab} in the sense of measure: for almost all $t\in(0,T)$,

$$\mu_{ab}^h(t)\overset{*}{\rightharpoonup}\mu_{ab}(t)\quad\text{in the dual of } C_0^0(\Omega).\tag{5.4}$$

Then we use an argument similar to the one in the proof of the Reshetnyak Lemma (cf. [**32**]). For any $\varepsilon>0$, we find some smooth vector field g_ε which, on $\partial^*\Omega_a^h$, is a good approximation of v^h: for $a=1,\dots,N$, for almost all $t\in(0,T)$ and for all $\varepsilon>0$ there exists $g_\varepsilon\in(C_0^\infty(\Omega))^n$ with $|g_\varepsilon|\leqq 1$ and

$$\limsup_{h\to 0}\int_\Omega\left|g_\varepsilon-\frac{\nabla\chi_a^h}{|\nabla\chi_a^h|}\right||\nabla\chi_a^h|\leqq C\varepsilon^{\frac{1}{2}}.\tag{5.5}$$

Using this we finally show that for almost all $t\in(0,T)$ and all smooth vector fields ζ

$$\lim_{h\to 0}\int_\Omega v^h\cdot\nabla\zeta v^h\mu_{ab}^h=\int_\Omega v\cdot\nabla\zeta v\mu_{ab}.\tag{5.6}$$

Since ∇w_a^h converges weakly in L^2, we may thus pass to the limit $h\to 0$ in the Euler–Lagrange equation (5.3) and obtain (4.2) of the definition of a weak solution.

We now finish the proof by showing (5.4) to (5.6). To show (5.4) we first show $\int_\Omega f\mu_{ab}^h\to\int_\Omega f\mu_{ab}$ for all $f\in C^0(\bar\Omega)$. Since $\int_\Omega|\nabla\chi_a^h|\to\int_\Omega|\nabla\chi_a|$, we conclude that for all $f\in C^0(\bar\Omega)$ (cf. [22])

$$\int_\Omega f|\nabla\chi_a^h|\to\int_\Omega f|\nabla\chi_a|.$$

Furthermore, we know that (cf. the proof of Lemma 5.3)

$$\int_\Omega f|\nabla\chi_a^h|=\sum_{b=1,b\neq a}^N\int_\Omega f\mu_{ab}^h.$$

Using $\liminf_{h\to 0}\int_D|\nabla(\chi_a^h+\chi_b^h)|\geqq\int_D|\nabla(\chi_a+\chi_b)|$ for all open $D\in\Omega$ implies

$$\limsup_{h\to 0}\int_\Omega f\mu_{ab}^h\leqq\int_\Omega f\mu_{ab}$$

for all non-negative $f\in C^0(\bar\Omega)$.

Now assume that a strict inequality were to hold in the above. Then

$$\begin{aligned}\int_\Omega f|\nabla\chi_a| &= \lim_{h\to 0}\int_\Omega f|\nabla\chi_a^h|\\ &= \lim_{h\to 0}\int_\Omega\sum_{b=1,b\neq a}^N f\mu_{ab}^h\leqq\sum_{b=1,b\neq a}^N\limsup_{h\to 0}\int_\Omega f\mu_{ab}^h\\ &< \sum_{b=1,b\neq a}^N\int_\Omega f\mu_{ab}=\int_\Omega f|\nabla\chi_a|.\end{aligned}$$

We have a contradiction. Since the same argument applies to any subsequence, the claim (5.4) is shown.

From the definition of $\int_\Omega|\nabla\chi_a|$, we deduce that for all $\varepsilon>0$ there exists a smooth vector field g_ε with $|g_\varepsilon|\leqq 1$, such that

$$\int_\Omega g_\varepsilon\cdot\nabla\chi_a=-\int_\Omega\operatorname{div}g_\varepsilon\chi_a\,dx\geqq\int_\Omega|\nabla\chi_a|-\varepsilon.$$

Since $\nabla\chi_a^h$ converges in the weak-* topology and since for almost all t there is no loss of perimeter, we find

$$\lim_{h\to 0}\left(\int_\Omega|\nabla\chi_a^h|-\int_\Omega g_\varepsilon\cdot\nabla\chi_a^h\right)=\int_\Omega|\nabla\chi_a|-\int_\Omega g_\varepsilon\cdot\nabla\chi_a\leqq\varepsilon.$$

This implies

$$\lim_{h\to 0}\int_\Omega\left(1-g_\varepsilon\frac{\nabla\chi_a^h}{|\nabla\chi_a^h|}\right)|\nabla\chi_a^h|=\int_\Omega\left(1-g_\varepsilon\frac{\nabla\chi_a}{|\nabla\chi_a|}\right)|\nabla\chi_a|\leqq\varepsilon.$$

Now we use on $\partial^*\Omega_a$

$$\left|g_\varepsilon-\frac{\nabla\chi_a^h}{|\nabla\chi_a^h|}\right|^2=g_\varepsilon^2-2g_\varepsilon\frac{\nabla\chi_a^h}{|\nabla\chi_a^h|}+1\leqq 2\left(1-g_\varepsilon\frac{\nabla\chi_a^h}{|\nabla\chi_a^h|}\right)$$

and

$$\int_\Omega\left|g_\varepsilon-\frac{\nabla\chi_a^h}{|\nabla\chi_a^h|}\right||\nabla\chi_a^h|\leqq\left(\int_\Omega\left|g_\varepsilon-\frac{\nabla\chi_a^h}{|\nabla\chi_a^h|}\right|^2|\nabla\chi_a^h|\right)^{\frac12}\left(\int_\Omega|\nabla\chi_a^h|\right)^{\frac12}$$

to obtain (5.5).

Finally, we compute

$$\begin{aligned}&\int_\Omega v^h\cdot\nabla\zeta v^h\mu_{ab}^h-\int_\Omega v\cdot\nabla\zeta v\mu_{ab}\\ &=\int_\Omega(v^h-g_\varepsilon)\cdot\nabla\zeta(v^h+g_\varepsilon)\mu_{ab}^h+\int_\Omega g_\varepsilon\cdot\nabla\zeta g_\varepsilon(\mu_{ab}^h-\mu_{ab})+\int_\Omega(v-g_\varepsilon)\cdot\nabla\zeta(v+g_\varepsilon)\mu_{ab}.\end{aligned}$$

Since $\mu^h_{ab} \overset{*}{\to} \mu_{ab}$, and since $|\nabla \chi^h_a| = \Sigma^N_{b=1, b\neq a} \mu^h_{ab}$ and $\nu^h = \nabla \chi^h_a / |\nabla \chi^h_a|$ with respect to the measure μ^h_{ab}, we may apply (5.4) and (5.5) to deduce (5.6). $\square$

6. Conclusion

We have extended the Mullins–Sekerka model for phase-separation in binary systems to a multi-phase model including triple junctions. Using formal asymptotic expansions, we have related this sharp interface model to a transition layer model known as the Cahn–Hilliard system. We then discussed some geometric properties of the multi-phase Mullins–Sekerka flow.

We proposed a weak formulation based on integration by parts on manifolds and we showed that smooth weak solutions satisfy the strong equations. We introduced an implicit time discretisation using energy minimisation in each time step. Eventually we obtained a conditional existence result.

Appendix

In this appendix, we follow the argument of Bronsard and Reitich [5] and derive the angle condition which must be satisfied at a triple junction. In the moving rescaled coordinate y introduced in the fourth part of Section 2, the scaled versions of the curves Γ_{ab} agree to first order with their tangential half lines T_{ab}. On T_{ab} we have the coordinate system ν_{ab} and τ_{ab}, and we assume the latter to point away from the triple point. We note that these coordinate systems not only exist on T_{ab} but may naturally be extended into all of $\mathbf{R}^2$. Without loss of generality, we assume that τ_{jk} is the negative of the second standard unit vector e_2.

We consider a big equilateral triangle Δ with centre at the origin (which is the rescaled triple point), whose edges $\partial_{ab}\Delta$ intersect the tangent lines T_{ab}. We assume that $\partial_{jk}\Delta$ is perpendicular to T_{jk}, while the other two edges may intersect the other two tangent lines at an arbitrary angle. We denote the outer normal of Δ by $n = (n_1, n_2)$.

We multiply the differential equation for $\mathscr{U}^0$ by $\partial_{\nu_{jk}} \mathscr{U}^0$, integrate over the triangle and use the Divergence Theorem:

$$\int_\Delta \partial_{\nu_{jk}} \psi(\mathscr{U}^0)\, dy = \int_\Delta \Delta_y \mathscr{U}^0 \cdot \partial_{\nu_{jk}} \mathscr{U}^0 \, dy$$

$$\Leftrightarrow \int_\Delta \partial_{\nu_{jk}} \left(\psi(\mathscr{U}^0) + \frac{1}{2}(|\partial_{\tau_{jk}} \mathscr{U}^0|^2 - |\partial_{\nu_{jk}} \mathscr{U}^0|^2) \right) dy = \int_\Delta \partial_{\tau_{jk}} (\partial_{\nu_{jk}} \mathscr{U}^0 \cdot \partial_{\tau_{jk}} \mathscr{U}^0)\, dy$$

$$\Leftrightarrow \int_{\partial\Delta} \left(\psi(\mathscr{U}^0) + \frac{1}{2}(|\partial_{\tau_{jk}} \mathscr{U}^0|^2 - |\partial_{\nu_{jk}} \mathscr{U}^0|^2) \right) n_1 \, do = - \int_{\partial\Delta} (\partial_{\nu_{jk}} \mathscr{U}^0 \cdot \partial_{\tau_{jk}} \mathscr{U}^0)\, n_2 \, do.$$

To evaluate these integrals, we use the matching conditions. Indeed, let θ_k and θ_j be the angle between ν_{jk} and ν_{kl} and ν_{jk} and ν_{lj}, respectively. We note that on the edge $\partial_{kl}\Delta$ we have $n_1 = \cos 30°$ and $n_2 = \sin 30°$, and, if the triangle is sufficiently big, the matching condition implies that

$$\partial_{\nu_{jk}} \mathscr{U}^0(y) \approx (\cos\theta_k) \partial_{\nu_{kl}} U^0_{kl}(y \cdot \nu_{kl}) \quad \text{and} \quad \partial_{\tau_{jk}} \mathscr{U}^0(y) \approx -(\sin\theta_k) \partial_{\nu_{kl}} U^0_{kl}(y \cdot \nu_{kl}),$$

where U^0_{kl} is the one-dimensional stationary wave solution connecting phase $A(k)$ to phase $A(l)$. Similar relations hold on the other edges of the triangle.

Substituting this into the above identity gives

$$\int_{\partial_{kl}\Delta}\left(\psi(U^0_{kl})+\frac{1}{2}((\sin\theta_k)^2-(\cos\theta_k)^2)\,|\partial_{v_{kl}}U^0_{kl}|^2\right)\cos 30^\circ\,do$$

$$+\int_{\partial_{lj}\Delta}\left(\psi(U^0_{lj})+\frac{1}{2}((\sin\theta_j)^2-(\cos\theta_j)^2)\,|\partial_{v_{lj}}U^0_{lj}|^2\right)(-\cos 30^\circ)\,do$$

$$\approx\int_{\partial_{kl}\Delta}\cos\theta_k\sin\theta_k\,|\partial_{v_{kl}}U^0_{kl}|^2\sin 30^\circ\,do-\int_{\partial_{lj}\Delta}\cos\theta_j\sin\theta_j\,|\partial_{v_{lj}}U^0_{lj}|^2\sin 30^\circ\,do.$$

The edge $\partial_{kl}\Delta$ forms with v_{kl} an angle $\alpha_k := \theta_k - 120^\circ$. Thus for any function f that only depends on the coordinate z in direction v_{kl}, we have

$$\int_{\partial_{kl}\Delta} f\,do=\frac{1}{\cos\alpha_k}\int_{T^\perp_{kl}} f\,dz.$$

The same applies for $\partial_{lj}\Delta$, which forms the angle $\alpha_j := \theta_j - 120^\circ$ with $T^\perp_{lj}$.

Since the potential is symmetric, the energy of any one-dimensional stationary wave is the same, and due to the equipartition of energy

$$\frac{1}{2}\sigma=\frac{1}{2}\sigma_{ab}\approx\int_{T^\perp_{ab}}\psi(U^0_{ab})\,dz=\int_{T^\perp_{ab}}\frac{1}{2}|\partial_z U^0_{ab}|^2\,dz.$$

Thus we get

$$\frac{\cos 30^\circ}{\cos\alpha_k}\sin^2\theta_k-\frac{\cos 30^\circ}{\cos\alpha_j}\sin^2\theta_j=\frac{\sin 30^\circ}{\cos\alpha_k}\cos\theta_k\sin\theta_k-\frac{\sin 30^\circ}{\cos\alpha_j}\cos\theta_j\sin\theta_j.$$

This simplifies into

$$\sin\theta_k=\sin\theta_j.$$

Of course we obtain the same identity for any other pair (k, l) and (l, j). Thus we may determine the angles and obtain the angle condition (ac^0). But although we get three formulae, they are not independent, and we still have to use that the three angles have to add up to 360°. As a consequence of the angle condition, the normal vectors add up to 0.

In the nonsymmetric case we get similar formulae.

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