Curvature driven interface evolution

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Abstract

Curvature driven surface evolution plays an important role in geometry, applied mathematics and in the natural sciences. In this paper geometric evolution equations such as mean curvature flow and its fourth order analogue motion by surface diffusion are studied as examples of gradient flows of the area functional. Also in many free boundary problems the motion of an interface is given by an evolution law involving curvature quantities. We will introduce the Mullins-Sekerka flow and the Stefan problem with its anisotropic variants and discuss their properties.

In phase field models the area functional is replaced by a Ginzburg-Landau functional leading to a diffuse interface model. We derive the Allen-Cahn equation, the Cahn-Hilliard equation and the phase field system as gradient flows and relate them to sharp interface evolution laws.

1 Introduction

The motion of hypersurfaces in Euclidean space driven by a law for the normal velocity which involves curvature quantities plays an important role in geometry, analysis and in many applications. The most prominent example is the *mean curvature flow*, where the normal velocity of a hypersurface is given as the mean curvature of the surface. This evolution law appears in geometry but variants of this flow have applications in image processing and they also describe the evolution of so-called grain boundaries in materials science.

We will explain later that mean curvature flow in some sense is the most efficient way to decrease the area of a surface, in particular mean curvature flow turns out to be a gradient flow of the area functional. For surfaces that bound a region mean curvature flow typically decreases the enclosed volume. If one wants to preserve the enclosed volume one could study a volume conserving *nonlocal mean curvature flow*. A flow with the same property which has more physical applications is the *surface diffusion flow*. Here, the normal velocity is given by minus the surface Laplacian of the mean curvature.

In some physical systems the surface energy of an interface is proportional to the total surface area of the interface. This is true for example for soap bubbles and related variational problems lead to the geometry and analysis of minimal

surfaces and H-surfaces. We refer to the beautiful book by Hildebrandt and Tromba [97] and to [49], [119] for details.

We will also study the evolution of surfaces bounding a crystal. For crystals the surface area is not the appropriate energy. Instead the surface contribution to the energy will locally depend on the orientation of the surface in its surrounding space. The resulting surface energy will be anisotropic and related variational problems will lead to surfaces for which an anisotropic mean curvature will be either zero or constant. It is of course possible to consider gradient flows of such energies and this will lead us to the anisotropic mean curvature flow.

Often the evolution of a surface is influenced by quantities which are defined away from the surface. We will discuss the growth of a crystal and in this case the evolution of the crystal surface will be influenced for example by the temperature. In fact one has to solve a heat equation for the temperature away from the surface and in some models the temperature enters the mean curvature flow equation as an additional right-hand side.

Typically the topology of the surface will change during the evolution. If this happens a classical description of the surface involving parametrizations will develop singularities and hence will break down. We will discuss two approaches which will allow to pass through singularities. The first one involves Caccioppoli sets which are sets for which the characteristic function is of bounded variation. The second one is the phase field approach which describes the interface with the help of a smooth function which in an appropriate way approximates the characteristic function mentioned above. Another popular approach which is suitable to deal with topology changes is the level set method and we refer to [39, 69, 84, 130, 147].

In this overview article I can of course only describe a few aspects of curvature driven interface evolution. I will focus on mean curvature flow, its fourth order analogue motion by surface diffusion and on crystal growth described by the Stefan problem with Gibbs–Thomson law. The latter is a paradigm free boundary problem. In a free boundary problem one seeks a solution of a partial differential equation on a domain which one has to find as part of the problem.

Curvature driven interfaces play an important role also in other areas. In geometry one is also interested in situations where the evolution of a surface is driven by laws involving other curvature quantities such as the Gauss curvature, the scalar curvature or quantities involving the principal curvatures. Let me finally also mention applications in which curvature driven interface evolution plays a role. Examples are two-phase and free surface flow [137, 151], image analysis [7, 31, 32, 143], grain boundary motion [123], quantum dot formation [152], evolution of nanoporosity in dealloying [56], void evolution in electromigration [42], and flame propagation [148]. This list demonstrates that it is important both from a mathematical and from an applicational point of view to understand curvature driven interface evolution.

In order to illustrate the evolution laws discussed in this article we will fre-

quently use numerical computations which were obtained with the help of parametric finite element methods which have been developed in the last years together with Barrett and Nürnberg [8]-[13].

2 Gradient flows of the area functional

2.1 First variation of the area functional

We consider a smooth, compact, oriented hypersurface Γ in $\mathbb{R}^d$ without boundary. The simplest surface energy of such a hypersurface Γ is proportional to the surface area of Γ. We hence consider the area functional

$$E(\Gamma) := \mathcal{H}^{d-1}(\Gamma) \tag{2.1}$$

where $\mathcal{H}^{d-1}$ is the $(d-1)$–dimensional surface measure. The goal now is to evolve Γ in such a way that the surface area decreases most rapidly. Roughly speaking this will be achieved by flowing Γ in the direction of the negative "gradient" of E. In order to define the gradient we first of all need to determine the first variation (the "derivative") of the area functional.

In order to compute a directional derivative of E we need to embed Γ in a one-parameter family of surfaces. This will be achieved with the help of a smooth vector field $\zeta : \mathbb{R}^d \to \mathbb{R}^d$. We define

$$\Gamma_t := \{x + t\zeta(x) \mid x \in \Gamma\}\,, \quad t \in \mathbb{R}\,, \tag{2.2}$$

and a computation, see e.g. [47], [91], [111], gives

$$\frac{d}{dt} E(\Gamma_t)_{|t=0} = -\int_\Gamma HV\, d\mathcal{H}^{d-1}\,. \tag{2.3}$$

Here H is the mean curvature of Γ (which in this article will be, as often in the literature, the sum of the principal curvatures), $V = \zeta \cdot \nu$ is the normal velocity of the evolving surface $(\Gamma_t)_{t\in\mathbb{R}}$ at $t = 0$, $\zeta \cdot \nu$ is the Euclidean inner product of ζ and ν, and by $d\mathcal{H}^{d-1}$ we denote integration with respect to the $(d-1)$–dimensional surface measure. On Γ we have chosen a normal vector field ν and we here take the sign convention that the surface has positive mean curvature if it is curved in the direction of the normal. The formula (2.3) now shows that the surface area decreases if the surface moves in the direction of the mean curvature vector $H\nu$.

2.2 Gradient flows

For a function $\Phi : \mathbb{R}^n \to \mathbb{R}$ with derivative $d\Phi_{x_0}$ at the point $x_0 \in \mathbb{R}^n$ we define the gradient $\operatorname{grad}\Phi(x_0) \in \mathbb{R}^n$ such that the following identity holds

$$d\Phi_{x_0}(v) = (\operatorname{grad}\Phi(x_0)) \cdot v \quad \text{ for all } v \in \mathbb{R}^n\,.$$

Now $x : [0,T] \to \mathbb{R}^n$ is a solution of the gradient flow equation to Φ if

$$x'(t) = -\text{grad}\,\Phi(x(t)) \tag{2.4}$$

holds for all $t \in [0,T]$. In particular, we have

$$\frac{d}{dt}\Phi(x(t)) = -\|\text{grad}\,\Phi(x(t))\|^2 \leq 0$$

where $\|\,.\,\|$ denotes the Euclidean norm in $\mathbb{R}^n$. In particular, we obtain that $\Phi(x(t))$ can only decrease in time.

For any $y : [0,T] \to \mathbb{R}^n$ with $\|y'(0)\| = \|\text{grad}\,\Phi(x(0))\|$ and $y(0) = x(0)$ we have, using the Cauchy-Schwarz inequality

$$\begin{aligned}\frac{d}{dt}\Phi(y(0)) &= (\text{grad}\,\Phi(y(0))) \cdot y'(0) \\ &= (\text{grad}\,\Phi(x(0))) \cdot y'(0) \\ &\geq -\|\text{grad}\,\Phi(x(0))\|^2\end{aligned}$$

with an equality if and only if

$$y'(0) = -\text{grad}\,\Phi(x(0))\,.$$

This shows that among all possible directions, the direction $-\text{grad}\,\Phi(x(0))$ decreases Φ most efficiently.

The above considerations can be generalized to n–dimensional Riemannian manifolds M and functions $\Phi : M \to \mathbb{R}$. Denoting by T_xM the tangent space at $x \in M$, by $\langle\,.\,,.\,\rangle$ the metric on T_xM and by $d_x\Phi$ the differential of Φ, the gradient $\text{grad}_M\Phi \in T_xM$ is defined such that

$$d_x\Phi(v) = \langle \text{grad}_M\Phi(x), v\rangle \quad \text{ for all } v \in T_xM$$

holds. Hence, as above the flow

$$x'(t) = -\text{grad}_M\Phi(x(t))$$

decreases Φ as fast as possible among all velocities with a given value for the norm of the velocity.

Choosing a time step $\tau > 0$, a natural approximation scheme for (2.4) would be to solve iteratively for $x^1, x^2, x^3 \ldots$ with a given initial value x^0 the (nonlinear) equation

$$\frac{x^n - x^{n-1}}{\tau} = -\text{grad}\,\Phi(x^n), \quad n = 1,2,3,\ldots \tag{2.5}$$

which is an implicit Euler discretization for (2.4). The identity (2.5) is the Euler–Lagrange equation of the functional

$$\tfrac{1}{2\tau}\|x - x^{n-1}\|^2 + \Phi(x)\,. \tag{2.6}$$

Now a natural approach to show existence of solutions to (2.5) is the study of the minimum problem to (2.6) which often can be solved by the direct method of the calculus of variations.

The approach can be naturally generalized to Hilbert spaces by replacing the Euclidean inner product by a general scalar product. It is also possible to generalize the scheme to metric spaces by replacing the norm of $x - x^{n-1}$ by the distance between x and x^{n-1}, i.e. one now considers

$$\tfrac{1}{2\tau} d^2(x, x^{n-1}) + \Phi(x)\,.$$

We refer to Ambrosio, Gigli and Savaré [3] and Villani [160] for more details on the general gradient flow approach and to Luckhaus [109], Visintin [161], Almgren, Taylor, Wang [5], Luckhaus, Sturzenhecker [110], Mielke, Theil, Levitas [116], Otto [131] and Garcke, Schaubeck [79] for applications of the approach in specific situations.

2.3 Mean curvature flow as a gradient flow of the area functional

We formally endow the space $\mathcal{M}$ of all oriented hypersurfaces Γ in $\mathbb{R}^d$ with a tangent space which consists of all possible normal velocities, i.e. we set

$$T_\Gamma \mathcal{M} = \{V : \Gamma \to \mathbb{R}\}\,.$$

A function $V : \Gamma \to \mathbb{R}$ arises as a "tangent" vector, i.e. as a differential of a curve in $\mathcal{M}$, if we consider a vector field $\zeta : \mathbb{R}^d \to \mathbb{R}^d$ such that $\zeta \cdot \nu = V$ on Γ and define Γ_t as in (2.2). One natural choice of an inner product on $T_\Gamma \mathcal{M}$ is given by

$$\langle v_1, v_2 \rangle_{L^2} = \int_\Gamma v_1 v_2 \, d\mathcal{H}^{d-1} \quad \text{for all } v_1, v_2 \in T_\Gamma \mathcal{M}\,.$$

Now the gradient $\operatorname{grad}_{\mathcal{M}} E$ of E needs to fulfill

$$\langle \operatorname{grad}_{\mathcal{M}} E, V \rangle_{L^2} = \frac{d}{dt} E(\Gamma_t)_{|t=0} = -\int_\Gamma H V d\mathcal{H}^{d-1}$$

for all $V : \Gamma \to \mathbb{R}$. Here $(\Gamma_t)_{t\in\mathbb{R}}$ is defined as above by choosing a $\zeta : \mathbb{R}^d \to \mathbb{R}^d$ such that $\zeta \cdot \nu = V$. We hence obtain

$$\operatorname{grad}_{\mathcal{M}} E = -H$$

and the gradient flow of the area functional E is the mean curvature flow

$$V = H\,.$$

More precisely, we say that a smooth one–parameter family $(\Gamma_t)_{t\geq 0}$ of hypersurfaces in $\mathbb{R}^d$ solves $V = H$ if for a local parametrization $X(t,p)$, $p \in U$, $U \subset \mathbb{R}^{d-1}$ open, it holds that

$$\partial_t X \cdot \nu = H\,.$$

In particular, we obtain

$$\frac{d}{dt}\mathcal{H}^{d-1}(\Gamma_t) = -\int_{\Gamma_t} H^2 \leq 0\,.$$

For more information on mean curvature flow we refer to the articles by Ecker [55] and White [168] and the books [22, 54, 84, 103, 111, 139].

Let me mention a few fundamental properties and results related to the mean curvature flow.

- An embedded curve in the plane evolving under curvature flow $V = H$ will become convex in finite time, see Grayson [92].

- A convex hypersurface in $\mathbb{R}^d$, i.e. a surface which is the boundary of a convex region, will shrink to a point in finite time. In doing so the surface will become more and more round, i.e. after rescaling to a surface enclosing a fixed volume the surface will converge to a sphere, see Gage, Hamilton [75] and Huisken [98].

- Mean curvature flow, written for example in local coordinates, leads to a parabolic equation of second order (the normal velocity leads to a time derivative and the mean curvature to two spatial derivatives). Second order parabolic partial differential equations fulfill *maximum and comparison principles* which play a fundamental role in the analysis of mean curvature flow. With the help of a comparison principle it can be shown that self-intersections during the flow are not possible and one can also show that if initially one surface is contained in another this property will be true for all later times, see e.g. the discussion in Ecker [55].

- Nonconvex surfaces in general can develop singularities, see Figures 1, 2 for numerical computations with a torus as initial surface. Depending on the ratio of the torus's radii, the torus will either merge, see Figure 1 or shrink to a circle, see Figure 2. The possible singularities are well understood and in particular the famous monotonicity formula for mean curvature flow is important in order to classify the singularities, see [55, 100, 101, 111].

Mean curvature flow does not preserve the volume enclosed by the surface. In many applications physical conservation laws lead to volume conservation and this motivates the discussion of volume preserving geometric gradient flows which are discussed in the next subsections.

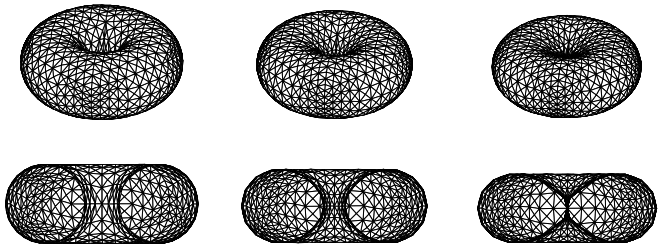


Figure 1: Plots of a solution to mean curvature flow at times $t = 0,\ 0.05,\ 0.09$.

2.4 Volume preserving geometric flows

We now consider $\mathcal{M}_m$ to be the "manifold" of all smooth hypersurfaces $\Gamma \subset \mathbb{R}^d$ enclosing a bounded set $G \subset \mathbb{R}^d$ which has a prescribed volume $m \in \mathbb{R}^+$. It will turn out that the natural tangent space $T_\Gamma \mathcal{M}_m$ corresponds formally to all normal velocities $V : \Gamma \to \mathbb{R}$ with zero mean. If we choose perturbations $(\Gamma_t)_{t\in\mathbb{R}}$ of Γ as above we obtain for the enclosed volume $vol(\Gamma_t)$ the following identity, see [47],

$$\frac{d}{dt}\mathrm{vol}(\Gamma_t) = \int_{\Gamma_t} V d\mathcal{H}^{d-1}\,,$$

where we now choose ν as the outer unit normal to the set enclosed by the hypersurface Γ. This implies that the integral of the normal velocity needs to be zero to ensure that the volume is conserved. We can also endow $T_\Gamma \mathcal{M}_m$ with the L^2–inner product. The gradient $\mathrm{grad}_{\mathcal{M}_m} E$ has to fulfill $\int_\Gamma (\mathrm{grad}_{\mathcal{M}_m} E)\, d\mathcal{H}^{d-1} = 0$ and

$$\langle \mathrm{grad}_{\mathcal{M}_m} E, v\rangle = -\int_\Gamma H v d\mathcal{H}^{d-1} \tag{2.7}$$

for all v with $\int_\Gamma v\, d\mathcal{H}^{d-1} = 0$. The identity (2.7) does specify $\mathrm{grad}_{\mathcal{M}_m} E$ only up to a constant and since $\mathrm{grad}_{\mathcal{M}_m} E$ needs to have zero mean we obtain

$$\mathrm{grad}_{\mathcal{M}_m} E = -H + \fint_\Gamma H d\mathcal{H}^{d-1}$$

where $\fint_\Gamma H d\mathcal{H}^{d-1} = \int_\Gamma H d\mathcal{H}^{d-1}/(\int_\Gamma 1\, d\mathcal{H}^{d-1})$ is the average of H on Γ. The volume preserving mean curvature flow is hence given as

$$V = H - \fint_\Omega H d\mathcal{H}^{d-1}\,. \tag{2.8}$$

A more physical gradient flow is obtained when we choose an H^{-1}–inner product on $T_\Gamma \mathcal{M}_m$. For given $v_1, v_2 \in T_\Gamma \mathcal{M}_m$ we solve

$$-\Delta_\Gamma u_i = v_i \quad \text{ on } \Gamma\,,$$

where Δ_Γ is the surface Laplacian on Γ. Since Γ has no boundary the Gauss theorem on manifolds gives $\int_\Gamma \Delta_\Gamma u_i\, d\mathcal{H}^{d-1} = 0$ which implies the solvability condition

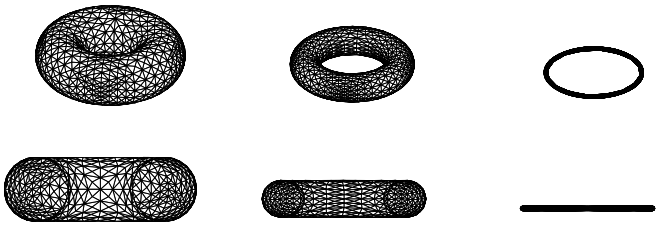


Figure 2: Plots of a solution to mean curvature flow at times $t = 0,\ 0.1,\ 0.138$.

$\int_\Gamma v_i \, d\mathcal{H}^{d-1} = 0$ which is fulfilled due to $v_i \in T_\Gamma \mathcal{M}_m$. Setting

$$u_i := (-\Delta_\Gamma)^{-1} v_i$$

we define the H^{-1}–inner product on $T_\Gamma \mathcal{M}_m$ as

$$\begin{aligned} \langle v_1, v_2 \rangle_{H^{-1}} &:= \int_\Gamma (\nabla_\Gamma (-\Delta_\Gamma)^{-1} v_1) \cdot (\nabla_\Gamma (-\Delta_\Gamma)^{-1} v_2) \, d\mathcal{H}^{d-1} \\ &= \int_\Gamma v_1 (-\Delta_\Gamma)^{-1} v_2 \, d\mathcal{H}^{d-1} \,. \end{aligned}$$

In order to define the gradient $\mathrm{grad}_{H^{-1}} E$ of E with respect to the H^{-1}–inner product we observe that the following identities need to hold for all $v \in T_\Gamma \mathcal{M}_m$

$$\begin{aligned} \int_\Gamma v (-\Delta_\Gamma)^{-1} \mathrm{grad}_{H^{-1}} E \, d\mathcal{H}^{d-1} &= \langle v, \mathrm{grad}_{H^{-1}} E \rangle_{H^{-1}} \\ &= -\int_\Gamma v H \, d\mathcal{H}^{d-1} \,. \end{aligned}$$

We hence obtain

$$\mathrm{grad}_{H^{-1}} E = \Delta_\Gamma H$$

and the H^{-1}–gradient flow of E is given as

$$V = -\Delta_\Gamma H \,. \tag{2.9}$$

This evolution law is called motion by surface diffusion. In physics this evolution law is derived from mass conservation laws using appropriate constitutive assumptions, see [124, 45, 158]. It models phase transformation due to diffusion along the interface. This evolution law can be derived from the Cahn-Hilliard diffusion equation, see Section 5, when diffusion is restricted to an interfacial layer, see [30, 58], and this also motivates that the law (2.9) is called motion by surface diffusion.

PROPOSITION. 2.1. *Solutions* $(\Gamma_t)_{t \geq 0}$ *of (2.9) fulfill*

$$\begin{aligned} \frac{d}{dt} \mathrm{vol}(\Gamma_t) &= 0 \,, \\ \frac{d}{dt} \mathrm{Area}(\Gamma_t) &\leq 0 \,. \end{aligned}$$

Proof. The second property follows from the fact that (2.9) is the gradient flow of the area functional with respect to the H^{-1}–inner product. The fact that the flow is volume preserving follows from

$$\begin{aligned} \frac{d}{dt}\mathrm{vol}(\Gamma_t) &= \int_{\Gamma_t} V \, d\mathcal{H}^{d-1} \\ &= -\int_{\Gamma_t} \Delta_\Gamma H \, d\mathcal{H}^{d-1} = 0\,, \end{aligned}$$

where the last identity is a consequence of Gauss' theorem. □ □

In the evolution law $V = -\Delta_\Gamma H$ the surface Laplacian –a second order operator– acts on the mean curvature and hence the flow leads to a fourth order parabolic partial differential equation. For fourth order parabolic equations maximum and comparison principles are in general not true. Also a monotonicity formula for the flow $V = -\Delta_\Gamma H$ is not known. These two facts are among the reasons why much less is known for the flow $V = -\Delta_\Gamma H$ in comparison to the mean curvature flow.

In the following I state a few results known for the surface diffusion flow.

- Short time existence and uniqueness of classical solutions is known. Spheres are asymptotically stable under the flow in the following sense. For initial data sufficiently close to a sphere a global solution exists and the solution will converge to a (possible different) sphere, see Elliott, Garcke [59] for curves in the plane and Escher, Mayer, Simonett [66] for higher dimensions.

- The flow $V = -\Delta_\Gamma H$ defines an analytic semigroup with some interesting properties. The set of equilibria is not isolated and in order to show stability of spheres either center manifold theory, see Escher, Mayer, Simonett [66], or a generalized principle of linearized stability has to be used, see Prüss, Simonett, Zacher [136].

- In the plane it can be shown that if a simple closed curve evolving under $V = -\Delta_\Gamma H$ exists for all time, it necessarily has to converge to a sphere, see Elliott and Garcke [59]. We also refer to a recent result by Wheeler [167] who proved that closed curves with initial data close to a round circle in the sense that the L^2–perturbation of the curvature remains small exist for all time and converges exponentially fast to a circle. A similar result also holds in higher dimensions, see [166].

- In contrast to mean curvature flow self intersections are possible, as was conjectured by Elliott, Garcke [59] and shown by Giga, Ito [88] and Mayer, Simonett [114], see also Blatt [19] for some recent results.

- The surface diffusion flow does not preserve convexity, see Figure 3, which was shown by Giga, Ito [87], see also Blatt [19].

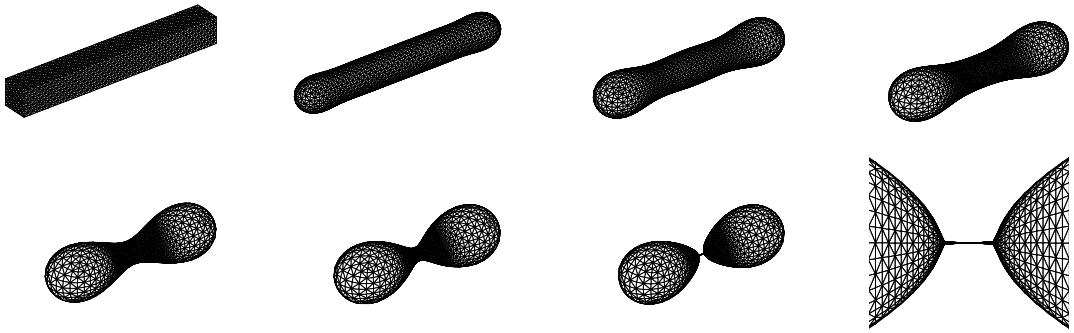


Figure 3: Plots of a solution to the surface diffusion flow at times $t = 0,\ 0.01,\ 0.1,\ 0.2,\ 0.3,\ 0.36,\ 0.369$ (left to right, top to bottom). The final plot shows a blow up of the pinch-off at time $t = 0.369$.

- A sharp criterion for finite time blow up of curves moving under the surface diffusion flow has been given by Chou [40].

- The sphere is stable under surface diffusion (see results above), while the cylinder is long wave unstable. Numerical simulations indicate that perturbations of a cylinder can lead to finite-time pinch-off. A paper by Bernoff, Bertozzi and Witelski [18] studies the selfsimilar structure close to the pinch-off.

- Initial data which are given as a graph can loose this property during the evolution, see [60].

- Since $V = -\Delta_\Gamma H$ leads to a parabolic equation, the flow has a regularizing effect, i.e. in particular edges and corners will become smooth during the flow. As for the mean curvature flow also the surface diffusion flow in general will develop singularities. The regularizing effect, the fact that surface diffusion does not preserve convexity and the formation of singularities can be observed in the numerical simulation in Figure 3.

2.5 The Mullins-Sekerka free boundary problem as a gradient flow of the area functional

We now consider a compact hypersurface Γ in $\mathbb{R}^d$ which separates two open sets Ω_- and Ω_+ in an open domain $\Omega \subset \mathbb{R}^d$, see Figure 4.

Let ν be the unit normal to Γ pointing into Ω_+ and n the outer unit normal to $\partial\Omega$. The Mullins-Sekerka free boundary problem describes the evolution of the spatial distribution of two phases (here the phases occupy the regions Ω_- and Ω_+) driven by the reduction of interfacial area and limited by diffusion.

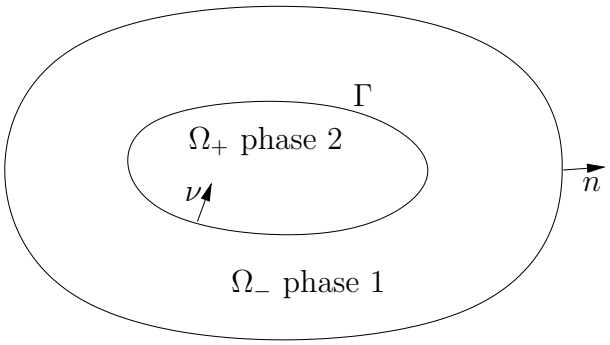


Figure 4: An illustration of the geometry in the Mullins-Sekerka and the Stefan problem.

This evolution law can be derived from conservation laws taking the principles of thermodynamics into account, see [94, 53]. But here we choose an approach which derives the Mullins-Sekerka problem in the context of gradient flows, see e.g. [70, 71, 113, 126].

We again consider the set of surfaces $\mathcal{M}_m$ which enclose a volume $m \in \mathbb{R}^+$ and its tangent space $T_\Gamma \mathcal{M}_m$. In order to define the metric on $T_\Gamma \mathcal{M}_m$ we define functions $u_1, u_2 : \Omega \to \mathbb{R}$ for given $v_1, v_2 \in T_\Gamma \mathcal{M}_m$ which are solutions of

$$\begin{aligned} -\Delta u_i &= 0 \quad \text{in } \Omega_- \cup \Omega_+ \,, && (2.10)\\ -[\nabla u_i]_-^+ \cdot \nu &= v_i \quad \text{on } \Gamma \,, && (2.11)\\ [u_i]_-^+ &= 0 \quad \text{on } \Gamma \,, && (2.12)\\ \nabla u_i \cdot n &= 0 \quad \text{on } \partial\Omega \,. && (2.13) \end{aligned}$$

Here $[\,.\,]_-^+$ denotes the jump of a quantity across the interface Γ where we subtract the value in the $-$-phase from the value in the $+$-phase. The above system (2.10)-(2.13) determines functions u_1, u_2 up to a constant which will be irrelevant for what follows. The metric $\langle .\,, . \rangle_{MS}$ on $T_\Gamma \mathcal{M}_m$ is now for all $v_1, v_2 \in T_\Gamma \mathcal{M}_m$ defined by

$$\langle v_1, v_2 \rangle_{MS} := \int_{\Omega_- \cup \Omega_+} \nabla u_1 \cdot \nabla u_2 dx = \int_\Gamma v_1 u_2 \, d\mathcal{H}^{d-1} \,,$$

where u_1, u_2 solve (2.10)-(2.13). We remark that the above system, which has to be solved in order to determine u_i, can be written in distributional form as

$$-\Delta u_i = v_i \delta_\Gamma \,, \tag{2.14}$$

where δ_Γ is a surface Dirac distribution defined by $\delta_\Gamma(\zeta) = \int_\Gamma \zeta$ for all test functions ζ. Hence formally $u_i = (-\Delta)^{-1}(v_i \delta_\Gamma)$ and hence $\langle .\,, . \rangle_{MS}$ can also be interpreted as an H^{-1}–inner product. We remark that a weak formulation of (2.14) is given as

$$\int_\Omega \nabla u_i \cdot \nabla \phi \, dx = \int_\Gamma v_i \phi \, d\mathcal{H}^{d-1}$$

which has to hold for all ϕ in the Sobolev space $H^1(\Omega) = \{f \in L^2(\Omega) \mid \nabla f \in L^2(\Omega, \mathbb{R}^d)\}$.

Now the negative gradient

$$w = -\mathrm{grad}_{MS} E \in T_\Gamma \mathcal{M}_m$$

needs to fulfill for all $v \in T_\Gamma \mathcal{M}_m$

$$\int_\Gamma vu \, d\mathcal{H}^{d-1} = \langle v, w \rangle_{MS} = \int_\Gamma vH \, d\mathcal{H}^{d-1}$$

where u is a solution to (2.10)-(2.13) with $v_i = w$.

In particular, we obtain that, up to an irrelevant constant, $u = H$ and hence the gradient flow $V = w = -\mathrm{grad}_{MS} E$ is for all $t > 0$ given as

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega_-(t) \cup \Omega_+(t)\,, && (2.15)\\ V &= -[\nabla u]^+_- \cdot \nu && \text{on } \Gamma_t\,, && (2.16)\\ u &= H && \text{on } \Gamma_t\,, && (2.17)\\ \nabla u \cdot n &= 0 && \text{on } \partial\Omega\,, && (2.18) \end{aligned}$$

where $\Omega_-(t)$, $\Omega_+(t)$ are the sets occupied by the two phases at time t.

PROPOSITION. 2.2. *Solutions* $((\Gamma_t)_{t\geq 0}, u)$ *to (2.15)-(2.18) fulfill*

$$\begin{aligned} \frac{d}{dt}\mathrm{vol}(\Omega_-(t)) &= 0\,, && (2.19)\\ \frac{d}{dt} Area(\Gamma_t) &= -\int_\Omega |\nabla u|^2 \leq 0 && (2.20) \end{aligned}$$

where $\mathrm{vol}(\Omega_-(t))$ *is the volume of* $\Omega_-(t) \subset \Omega$.

Proof. Although the area decrease follows from the gradient flow property we will show (2.20) directly. We have, using (2.15)-(2.18) and the Gauss theorem,

$$\begin{aligned} \frac{d}{dt} Area(\Gamma_t) &= -\int_{\Gamma_t} HV \, d\mathcal{H}^{d-1} = \int_{\Gamma_t} u\, [\nabla u]^+_- \cdot \nu \, d\mathcal{H}^{d-1}\\ &= -\int_{\Gamma_t} u(\nabla u^+) \cdot \nu^+ d\mathcal{H}^{d-1} - \int_{\Gamma_t} u(\nabla u^-) \cdot \nu^- d\mathcal{H}^{d-1}\\ &= -\int_{\Omega_+(t)} \mathrm{div}(\nabla u\, u) dx - \int_{\Omega_-(t)} \mathrm{div}(\nabla u\, u) dx = -\int_\Omega |\nabla u|^2 dx\,, \end{aligned}$$

where u^+ is u defined on $\Omega_+(t)$, u^- is u defined on $\Omega_-(t)$, $\nu^- = \nu$ is the outer unit normal to $\Omega_-(t)$ and $\nu^+ = -\nu$ is the outer unit normal to $\Omega_+(t)$. In addition, we have

$$\begin{aligned} \frac{d}{dt}\int_{\Omega_-(t)} 1\, dx &= \int_{\Gamma_t} V dx = -\int_{\Gamma_t} [\nabla u]^+_- \cdot \nu \, d\mathcal{H}^{d-1}\\ &= \int_{\Omega_+(t)} \mathrm{div}(\nabla u) dx + \int_{\Omega_-(t)} \mathrm{div}(\nabla u) dx = 0\,. \end{aligned}$$

□

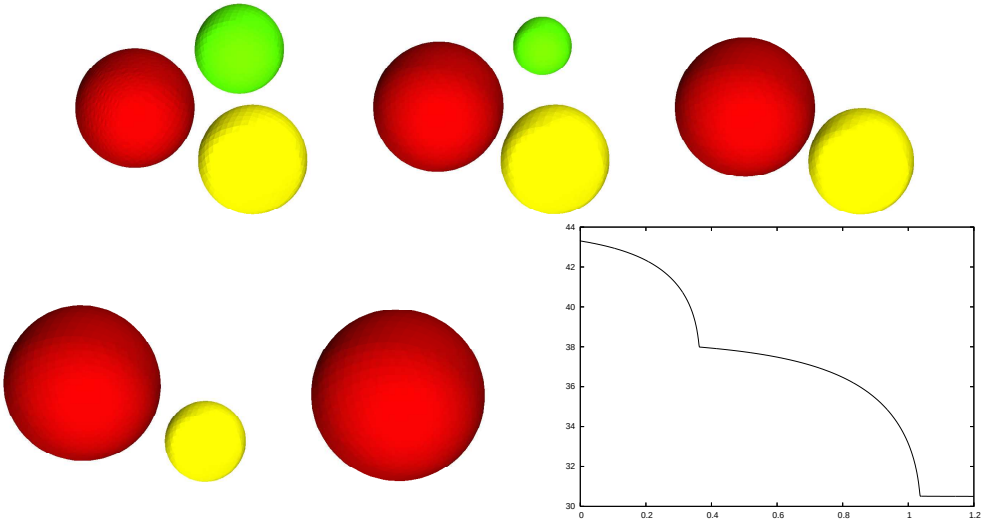


Figure 5: The solution of a Mullins-Sekerka problem with three particles at times $t = 0$, 0.3, 0.6, 0.9, 1.2 (left to right, top to bottom). On the bottom right, a plot of the total surface area as a function of time.

□

Mean curvature flow is a *second order* evolution equation and surface diffusion is a *fourth order* evolution equation. The operator which maps H to $[\nabla u]_-^+ \cdot \nu$ can be interpreted as a generalized *Dirichlet-to-Neumann operator* for the Laplace operator as it maps the Dirichlet data for the Laplace boundary value problem to Neumann data, see e.g. [62]. This operator is a nonlocal, pseudo-differential operator of first order and since the computation of the mean curvature already involves two spatial derivatives it turns out that the motion of the interface which separates the phases is determined by a nonlinear, nonlocal, pseudo-differential operator of *third order*, see [65].

The free boundary can also have different connected components and all the above is still valid as long as the components are disjoint. In particular, the overall volume of $\Omega_+(t)$ and $\Omega_-(t)$ is still preserved, see Figure 5. Now one effect involving different connected components is that typically small particles, i.e. small connected components, shrink and the energetically more advantageous large particles grow.

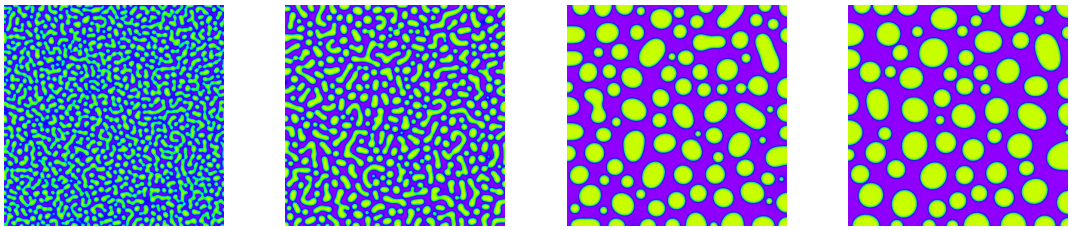
2.6 Results on the Mullins-Sekerka evolution

It is much more difficult to show existence of solutions to the Mullins-Sekerka problem in comparison to the geometric evolution equations discussed further above. This is due to the fact that the evolution of the hypersurface $(\Gamma_t)_{t\geq 0}$ determines the domains in which we have to solve Laplace equations and at the

same time the solutions of the Laplace equations determine the normal velocity of the evolving hypersurface. In order to be able to formulate the problem in a suitable setting involving appropriate function spaces one has to transform the time dependent domains $\Omega_+(t)$ and $\Omega_-(t)$, to fixed reference domains. This is done with the help of a diffeomorphism which depend on the solution $(\Gamma_t)_{t\geq 0}$ and was first introduced by Hanzawa [96]. Now the Laplace operator has to be transformed to the reference domain and altogether a highly nonlinear problem arises. If one formulates the transformed problem as an evolution equation for $(\Gamma_t)_{t\geq 0}$ one obtains a nonlinear, nonlocal, pseudo-differential operator of third order which carries a quasilinear structure. This approach was used independently by Escher, Simonett [64] and Chen, Hong, Yi [37] in order to show local existence of a unique local classical solution to (2.15)-(2.18) using a suitable contraction argument. Related earlier results in this direction are due to Duchon, Robert [51], Constantin, Pugh [41], Chen [35], who all showed existence in two spatial dimensions of problems related to (2.10)-(2.13).

Of course similar geometric questions as for the geometric evolutions above arise and we mention for example Mayer [112] who showed that the Mullins-Sekerka evolution does not preserve convexity of the enclosed domain.

The Mullins-Sekerka model describes the aging of materials which consist of a binary mixture of two components which are e.g. two different metals. In many of these systems two different phases with different concentrations form and the evolution of the boundaries of the phase regions is given by (2.15)-(2.18). Typically many particles appear and the total surface area is high, see Figure 6 (left).



(a) $t = 0$ (b) $t = 0.04$ (c) $t = 1.4$ (d) $t = 2.9$

Figure 6: Solutions to the Mullins-Sekerka problem reduce the total surface area by coarsening.

Now large particles grow, while smaller ones shrink and eventually vanish. Hence the number of particles decreases and typical length scales such as particle size and inter-particle distance increase, see Figure 6. This Ostwald ripening phenomenon has been studied intensively in the physics and metallurgical literature, see e.g. [162, 163] for reviews. For ensembles with a large number of particles, in which the volume fraction of one phase is small, it can be studied how the

mean particle size increases. In fact, the mean particle size will grow like $t^{\frac{1}{3}}$. In addition, evolution laws for the particle size distribution can be derived. We refer to [108, 125, 126, 127, 162, 163, 164, 104] for further details.

In general classical solutions to the Mullins-Sekerka problem do not exist for large times as topological changes and singularities can occur. For long-time existence results with general initial data one has to turn to weak formulations. Luckhaus and Sturzenhecker [110], see also [109], used a weak formulation of the identity $u = H$ in the setting of functions of bounded variations (BV-functions). The space of all functions of bounded variation is given as

$$BV(\Omega) = \left\{ f \in L^1(\Omega) \mid \textstyle\int_\Omega |\nabla f| < \infty \right\}$$

and $\int_\Omega |\nabla f|$ denotes the total variation of the distribution ∇f, i.e.

$$\int_\Omega |\nabla f| = \sup \left\{ \textstyle\int_\Omega f \operatorname{div} g \, dx \mid g \in C_0^1(\Omega, \mathbb{R}^d),\ |g(x)| \leq 1 \text{ for all } x \in \Omega \right\} .$$

For $f \in BV(\Omega)$ one obtains that ∇f and $|\nabla f|$ are Radon measures on Ω with values in $\mathbb{R}^d$ and $\mathbb{R}$ respectively. A measurable set $E \subset \Omega$ with $\int_\Omega |\nabla \chi_E| < \infty$, where χ_E is the characteristic function of E, is called Caccioppoli set. In a generalized sense such a set E has bounded perimeter. We can now define a generalized unit normal to the boundary of E given by $\nu_E = \frac{\nabla \chi_E}{|\nabla \chi_E|}$ as the Radon-Nikodym derivative of $\nabla \chi_E$ with respect to $|\nabla \chi_E|$. We refer to Giusti [91] and Ambrosio, Fusco, Pallara [2] for more details on functions of bounded variation.

The BV–formulation of Luckhaus and Sturzenhecker now replaces the pointwise identity $u = H$ by

$$\int_0^T \int_\Omega \left(\operatorname{div} \xi - \frac{\nabla \chi}{|\nabla \chi|} \cdot \left(D\xi \frac{\nabla \chi}{|\nabla \chi|} \right) \right) d|\nabla \chi(t)| dt = \int_{\Omega_T} \operatorname{div} (u\xi)\, \chi \, d(x,t) \,, \quad (2.21)$$

which has to hold for all $\xi \in C^1(\overline{\Omega}_T, \mathbb{R}^d)$, $\Omega_T := \Omega \times (0,T)$. Here $\chi : \Omega_T \to \{0,1\}$ is a phase function where phase 2 is given by the set $\{(x,t) \in \Omega \times (0,T) \mid \chi(x,t) = 1\}$ and phase 1 is given by the set $\{(x,t) \in \Omega \times (0,T) \mid \chi(x,t) = 0\}$ and one assumes that $\chi(.,t) \in BV(\Omega)$ for all $t \in (0,T)$.

If the interface is smooth and without boundary the equation (2.21) leads to

$$\int_0^T \int_{\Gamma_t} \operatorname{div}_\Gamma \xi \, d\mathcal{H}^{d-1} dt = - \int_0^T \int_{\Gamma_t} u \, \xi \cdot \nu \, d\mathcal{H}^{d-1} dt$$

and using the Gauss theorem on manifolds we have

$$\int_0^T \int_{\Gamma_t} H \xi \cdot \nu \, d\mathcal{H}^{d-1} dt = \int_0^T \int_{\Gamma_t} u \, \xi \cdot \nu \, d\mathcal{H}^{d-1} dt$$

which shows that (2.21) is a weak formulation of $u = H$.

Luckhaus and Sturzenhecker [110] used a time discretization of the Mullins-Sekerka problem which is similar to the discussion in Section 2.2, see (2.5), (2.6), to obtain approximate solutions. However they were only able to show that limits of this approximation solve the weak formulation of the Mullins-Sekerka problem under an additional assumption. They had to exclude a loss of surface area for the interface in the limit when a time discretization parameter tends to zero. We refer to Figure 7 for a situation in which there is a loss of area. The approach of Luckhaus and Sturzenhecker [110] was used for multi-phase situations in [80] and [23].

Later Röger [140] used methods from geometric measure theory and a result of Schätzle [145], who investigated the convergence of the equation $u_n = H_n$ in cases where u_n convergences as $n \to \infty$ in an approximate sense, to obtain a passage to the limit in time discrete approximations of the Mullins-Sekerka problem.

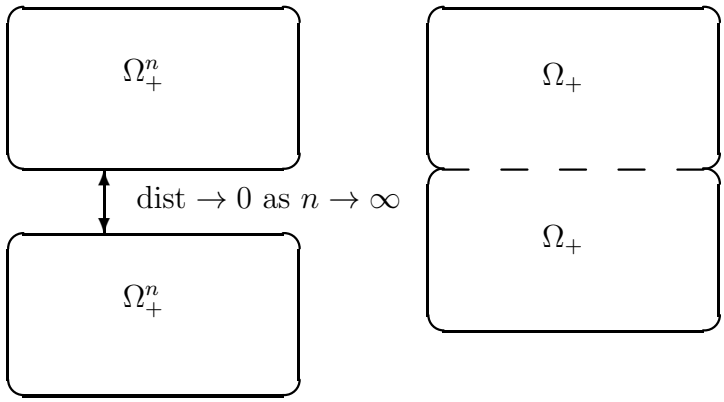


Figure 7: An example where a loss of area appears in a limit $n \to \infty$.

3 Anisotropic surface energies

3.1 Anisotropic variational problems and the Wulff shape

In many physical applications the surface energy density will depend on the local orientation of the surface in the surrounding space. Since we consider hypersurfaces, the local orientation can be expressed by a unit normal field $\nu : \Gamma \to \mathbb{R}^d$ ($|\nu| = 1$). We define the anisotropic surface energy as

$$E_\gamma(\Gamma) = \int_\Gamma \gamma(\nu) d\mathcal{H}^{d-1}$$

where

$$\gamma : \mathbb{R}^d \to [0, \infty)$$

is positively one-homogeneous, i.e. $\gamma(\lambda x) = \lambda\gamma(x)$ for all $\lambda > 0$, $x \in \mathbb{R}^d$. To define the energy E_γ it is enough to define γ on unit vectors but often it will be convenient to extend γ to all of $\mathbb{R}^d$ by requiring positive homogeneity of degree one. We remark that we obtain for $\gamma(x) = |x|$ the classical surface area. If γ is nonconstant on all unit normal vectors we have the phenomenon that some directions are energetically more favourable than others.

We can now consider a generalized isoperimetric problem for subsets $D \subset \mathbb{R}^d$ with smooth boundaries:

$$\text{minimize } \int_{\partial D} \gamma(\nu)\, d\mathcal{H}^{d-1} \qquad \text{subject to } \mathrm{vol}(D) = const. \tag{3.22}$$

It is known that the Wulff set

$$\mathcal{W}_\gamma := \{x \in \mathbb{R}^d \mid x \cdot y \leq \gamma(y) \text{ for all unit vectors } y \in \mathbb{R}^d\}$$

is the shape having the least anisotropic surface area among all sets enclosing the same volume. The problem (3.22) was formulated by Wulff [169] who also conjectured its solution. Dinghas [50] solved (3.22) among all convex polyhedra and later Taylor [155] and Fonseca, Müller [72, 73] gave existence and uniqueness proofs for very general interfacial energies.

It is helpful to visualize the interfacial energy density γ with the help of the Frank diagram

$$\mathcal{F}_\gamma = \{x \mid \gamma(x) \leq 1\}$$

which is the one-ball of γ. In Figure 8 we display the Frank diagram and the Wulff shape for a cubic and a hexagonal anisotropy. In these examples we observe the general fact that the Frank diagram and the Wulff shape are dual to each other, in the following sense. When γ is a convex, even function, it is a norm and has a dual norm. We obtain that $\mathcal{W}_\gamma$ is the unit ball of that dual norm, see [159, 16].

3.2 The first variation of anisotropic energies

For a given hypersurface Γ we now construct, as in Subsection 2.1, a one parametric family $(\Gamma_t)_{t\geq 0}$ with the help of a smooth vector field $\zeta : \mathbb{R}^d \to \mathbb{R}^d$. In order to compute the first variation of $\int_\Gamma \gamma(\nu)\, d\mathcal{H}^{d-1}$ we need the following ingredients:

i) $D\gamma(x) \cdot x = \gamma(x)$ for all $x \in \mathbb{R}^d \setminus \{0\}$,

ii) $D_t\nu = -\nabla_\Gamma V$,

iii) $\dfrac{d}{dt}\displaystyle\int_{\Gamma_t} f\, d\mathcal{H}^{d-1} = \int_{\Gamma_t} (D_t f - fHV) d\mathcal{H}^{d-1}$,

iv) $\displaystyle\int_{\Gamma_t} \nabla_\Gamma \cdot F\, d\mathcal{H}^{d-1} = -\int_{\Gamma_t} H\nu \cdot F\, d\mathcal{H}^{d-1}$ for $F \in C^1(\Gamma_t, \mathbb{R}^d)$.

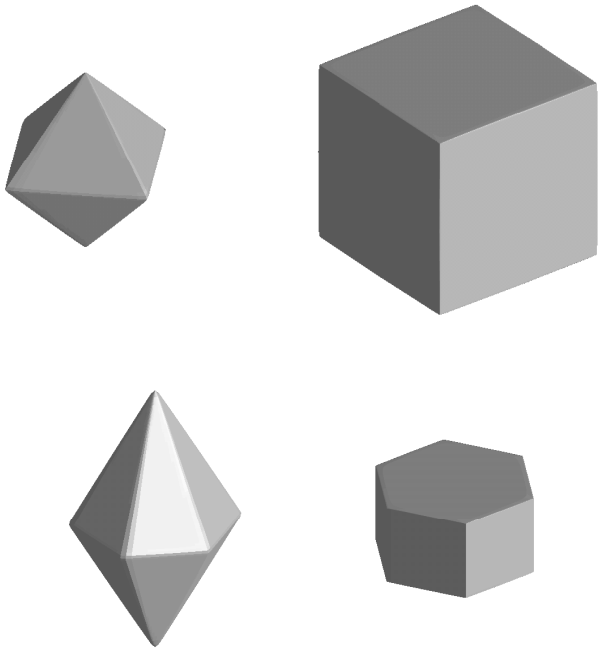


Figure 8: Frank diagrams (left) and Wulff shapes (right) for different choices of the anisotropic energy E_γ. Above we see a cubic anisotropy and below a hexagonal anisotropy, see [13] for details.

The first identity follows from the fact that γ is one-homogeneous. By $D_t f$ for a function f which is defined on an evolving surface we denote the normal time derivative of f, i.e. the time derivative following Γ_t with a velocity $V\nu$, see [93, formula (15-21)], and [81]. More precisely we choose a path $z(\tau) \in \Gamma_\tau$ such that $z'(\tau) = (V\nu)(\tau, z(\tau))$ and define $D_t f(t, z(t)) = \frac{d}{d\tau} f(\tau, z(\tau))$, $\tau = 1$. The identity (ii) is shown e.g. in [93, formula (15-24)], [55, 111]. For a proof of the transport theorem (iii) we refer to [47] or [81]. The equation iv) in the Gauss theorem on manifolds for vector fields F which are not necessarily tangential and a proof can be found e.g. in [90, Section 16]. We remark that for non-tangential vector fields $\nabla_\Gamma \cdot F$ is given as

$$\nabla_\Gamma \cdot F = \sum_{i=1}^{d-1} (\partial_{\tau_i} F) \cdot \tau_i$$

where $\{\tau_1, \dots, \tau_{d-1}\}$ is an orthonormal basis of the tangent space.

We now compute

$$\begin{aligned}
\frac{d}{dt}\int_{\Gamma_t}\gamma(\nu)d\mathcal{H}^{d-1} &= \int_{\Gamma_t}(D_t\gamma(\nu)-\gamma(\nu)HV)d\mathcal{H}^{d-1}\\
&= \int_{\Gamma_t}(D\gamma(\nu)\cdot D_t\nu-\gamma(\nu)HV)d\mathcal{H}^{d-1}\\
&= \int_{\Gamma_t}(-D\gamma(\nu)\cdot\nabla_\Gamma V-\gamma(\nu)HV)d\mathcal{H}^{d-1}\\
&= \int_{\Gamma_t}(\nabla_\Gamma\cdot(D\gamma(\nu))V+(D\gamma(\nu)\cdot\nu)VH-\gamma(\nu)HV)d\mathcal{H}^{d-1}\\
&= \int_{\Gamma_t}\nabla_\Gamma\cdot(D\gamma(\nu))V\,d\mathcal{H}^{d-1}\,.
\end{aligned}$$

Here we used i), ii), iii) and the Gauss theorem on manifolds, see iv).

Hence the negative L^2–gradient of E_γ is given by

$$H_\gamma=-\nabla_\Gamma\cdot(D\gamma(\nu))$$

which is known as the anisotropic mean curvature in the literature, see Taylor [157, 159]. Solutions of the classical isoperimetric problem have constant mean curvature. It turns out that solutions of the anisotropic version of the isoperimetric problem (3.22) lead to surfaces with constant anisotropic mean curvature.

3.3 Gradient flows of the anisotropic surface energy

We can replace the mean curvature H by the anisotropic mean curvature H_γ in all gradient flows studied in Section 2. We obtain in particular the anisotropic mean curvature flow

$$V=H_\gamma$$

and the anisotropic surface diffusion flow

$$V=-\Delta_\Gamma H_\gamma\,. \tag{3.23}$$

Both flows decrease the total anisotropic surface energy, i.e.

$$\frac{d}{dt}\int_{\Gamma_t}\gamma(\nu)d\mathcal{H}^{d-1}\le 0$$

and the latter flow will preserve the enclosed volume. We will hence expect that the flow will converge to an appropriately scaled Wulff shape. We refer to Figure 9 for a numerical computation with an anisotropy whose Wulff shape is a slightly regularized cube. We observe that for larger times the evolution will tend to the Wulff shape.

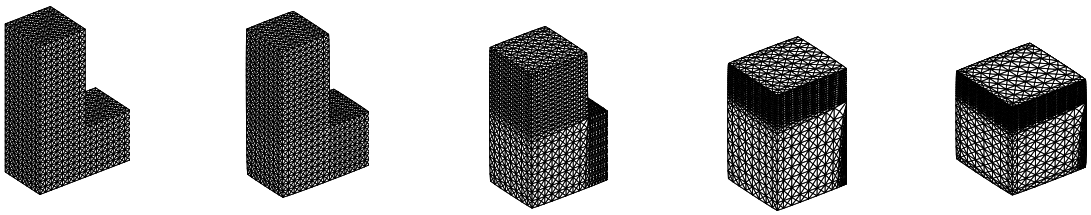


Figure 9: Plots of a solution to anisotropic surface diffusion (3.23) with a cubic anisotropy at times $t = 0,\ 0.01,\ 0.05,\ 0.1,\ 0.25$.

Often more general evolution laws of the form

$$\beta(\nu)V = H_\gamma$$

with a function $\beta : \mathbb{R}^d \to \mathbb{R}^+$ are of interest. Also these can be obtained as gradient flows by choosing an inner product of the form

$$\langle v_1, v_2 \rangle_\beta = \int_\Gamma \beta(\nu) v_1 v_2 \, d\mathcal{H}^{d-1} .$$

We refer e.g. to Bellettini and Paolini [17] who studied the case $\beta = \frac{1}{\gamma}$ which naturally arises when interpreting the anisotropic curvature flows in the context of Finsler geometry.

4 The Stefan problem

We have collected all ingredients to formulate the Stefan problem which describes solidification and melting phenomena in a general setting. The Stefan problem generalizes the Mullins-Sekerka problem studied in Section 3 as more physical effects are taken into account and in some sense it generalizes also the mean curvature flow as a forced mean curvature flow enters the overall system. In contrast to the Mullins-Sekerka problem where we solved Laplace's equation in the Stefan problem the heat equation has to be solved in the regions occupied by the phases and in the full Stefan problem also an additional time derivative can enter equation (2.17). We will not derive the Stefan problem from basic physical principles but refer to the books [94, 161] and [53] for a derivation.

The Stefan problem in a version taking anisotropic effects into account is now given as follows

$$\vartheta \partial_t u - \mathcal{K}_i \Delta u = 0 \qquad \text{in } \Omega_i(t), \text{ for } i \in \{-,+\}, \tag{4.24}$$

$$-[\mathcal{K}\nabla u]_-^+ \cdot \nu = \lambda V \qquad \text{on } \Gamma_t, \tag{4.25}$$

$$\beta(\nu)V = H_\gamma - au \qquad \text{on } \Gamma_t \tag{4.26}$$

together with appropriate initial and boundary conditions and non-negative physical constants $\vartheta, \mathcal{K}_-, \mathcal{K}_+, \lambda$ and a.

4.1 The classical Stefan problem

Typcially in the Stefan problem the unknown u describes the temperature in the system and the simplest modelling assumption for the temperature at the interface is that the temperature equals the melting temperature. Defining u to be the deviation from the melting temperature we have to choose $\beta = \gamma = 0$, $a \neq 0$ in (4.26) in order to obtain

$$u = 0 \tag{4.27}$$

as a boundary condition. The system (4.24), (4.25), (4.27) is called the classical Stefan problem and has been well-studied in the literature. We refer to the books by Elliott, Ockendon [61], Friedman [74], Meirmanov [115], Rubinstein [142] and Visintin [161] for more details, results and methods how to handle this problem analytically and numerically.

4.2 The Stefan problem with Gibbs–Thomson law

Setting $\beta = 0$, and for simplicity $a = 1$, (4.26) reduces to the Gibbs–Thomson law

$$u = H_\gamma \tag{4.28}$$

and the overall problem (4.24), (4.25), (4.28) is the Stefan problem with anisotropic Gibbs–Thomson law. An important contribution to this problem is due to Luckhaus [109]. He used an implicit time discretization, similar as the one discussed further above in the context of gradient flows, to show existence of a weak solution to (4.24), (4.25) together with

$$u = H \tag{4.29}$$

which is the isotropic version of the Gibbs–Thomson law. We now follow the lines of Rossi, Savaré [141] in order to describe the Stefan problem with Gibbs–Thomson law in the context of gradient flows. It is not so difficult to verify that the distributional formulation of (4.24), (4.25) is given by (for simplicity we set $\vartheta = \lambda = \mathcal{K}_- = \mathcal{K}_+ = 1$)

$$\partial_t(u + \chi) = \Delta u \tag{4.30}$$

where as before χ is the characteristic function of phase 2. Introducing the variable

$$e = u + \chi\,, \tag{4.31}$$

we can rewrite the equation (4.30) as

$$\partial_t e = \Delta(e - \chi)\,.$$

We now introduce

$$\Phi(e, \chi) = \int_\Omega (\tfrac{1}{2}|e - \chi|^2 + I_{\{0,1\}}(\chi))dx + \int_\Omega |\nabla \chi|$$

where the function $I_{\{0,1\}}$ is zero at 0 and 1 and ∞ elsewhere. The overall problem (4.24), (4.25), (4.29) can be written as

$$\begin{aligned} (-\Delta)^{-1}\partial_t e &= -(e-\chi)\,, \\ \Phi(e(t),\chi(t)) &\le \Phi(e(t),v) \quad \text{for all } v \in BV(\Omega,\{0,1\})\,. \end{aligned}$$

Rossi and Savaré [141] remark that this formulation naturally leads to the reduced functional

$$\phi(e) := \inf_{\chi} \Phi(e,\chi)$$

and one can formally consider the Stefan problem with Gibbs–Thomson law as the H^{-1}–gradient flow for ϕ in $H^{-1}(\Omega)$ as follows

$$(-\Delta)^{-1}\partial_t e = -\frac{\delta\phi}{\delta e}$$

where $\frac{\delta\phi}{\delta e}$ is the first variation of ϕ.

The anisotropic case is more involved and an existence result following the strategy of Luckhaus has been given in Garcke, Schaubeck [79], see also Kraus [105] for a slightly different approach. One crucial aspect is that the total variation has to be replaced by the anisotropic variation

$$\int_\Omega |\nabla f|_\gamma := sup\left\{ -\textstyle\int_\Omega f \operatorname{div}\varphi\, dx \mid \varphi \in C_0^1(\Omega,\mathbb{R}^d), \gamma^0(\varphi(x)) \le 1 \text{ a.e. } \right\},$$

where γ^0 is dual to γ, i.e.

$$\gamma^0(p) = \sup_{q\in\mathbb{R}^d\setminus\{0\}} \frac{p\cdot q}{\gamma(q)}\,.$$

In addition, an appropriate generalization of the weak formulation (2.21) has to be given, see [79].

4.3 Classical solutions to the Stefan problem

Local existence of classical solutions to the Stefan problem (4.24)-(4.26) can be shown by transforming the free boundary problem to a highly nonlinear problem on fixed domains. Similar as in the discussion at the beginning of Section 2.6 this approach uses the Hanzawa transformation. The problem on the fixed domain then has to be solved by a contraction argument using regularity theory for nonlinear parabolic equations. The isotropic case for the full Stefan problem (4.24)-(4.26) has been first considered by Radkevich [138] and Chen, Reitich [38].

Taking $\beta = 0$ in (4.26) leads to a certain quasi-static version of the Stefan problem and some new difficulties arise. Escher, Prüss and Simonett [67] were able to show local existence and uniqueness of analytic solutions in the isotropic case.

For the classical Stefan problem with $u = 0$ on Γ a comparison principle holds. For the Stefan problem with a curvature condition this is not possible any longer. Hence one has to come up with new ideas which replace methods which are based on comparison principles. Hadžić and Guo [95] were able to show stability of interfaces for the Stefan problem with curvature correction by developing a high-order energy method. We also mention a work by Prüss and Simonett [135] who study linear stability of spherical interfaces for (4.24)-(4.26) with $\beta > 0$.

Taking $\vartheta = 0$ leads to the Mullins-Sekerka problem. This problem in its anisotropic variant is relevant for many applications and also describes facetted growth of crystals. We only mention snow crystal growth as one example, see Figures 10, 11.

If the Wulff shape or the Frank diagram has flat parts or corners new difficulties arise as the anisotropic mean curvature is not well-defined any longer and the nonlocal crystalline mean curvature will replace the anisotropic mean curvature. This concept was introduced by Taylor [156] and Angenent, Gurtin [6]. Well-posedness problems for the resulting motions are quite difficult and issues like "facet bending" or "facet breaking" appear, see Figure 12 for an example of facet breaking. For more information on these problems we refer to the articles [82, 83, 86, 89, 14, 15, 120, 122]. Anisotropy, i.e. an orientation dependence through the normal, also appears in the kinetic coefficient β. The role played by the kinetic and the interfacial anisotropy in crystal growth is the subject of intense research. We refer to M.-H. Giga and Y. Giga [85] for details.

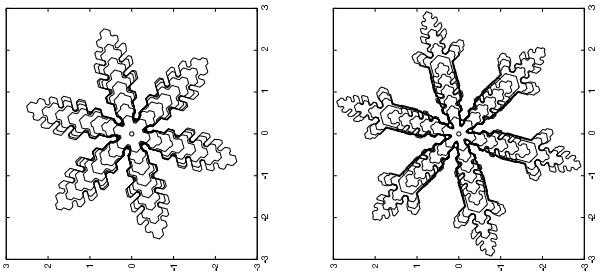


Figure 10: Two dimensional numerical computations of snow crystal growth, see [11, 12].

4.4 Numerical approaches

There have been many numerical approaches to geometrical evolution equations and free boundary problems, we refer to [47, 147] for reviews. The numerical computations presented in this paper have been developed in cooperation with John Barrett and Robert Nürnberg from the Imperial College in London. The

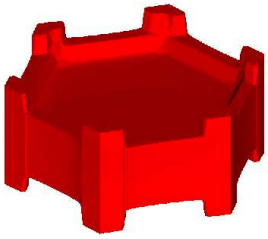


Figure 11: Also snow crystals like in this numerical computation are observed in nature and they are called columns on plates, see [12].

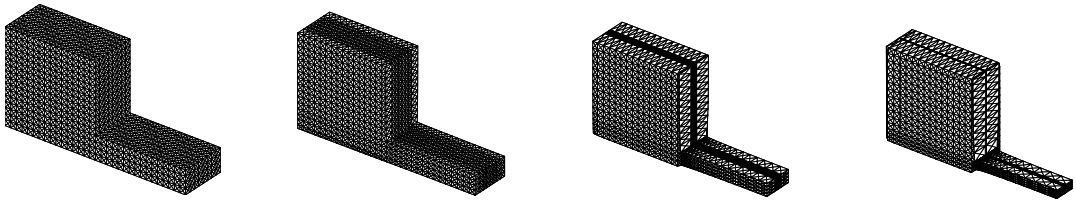


Figure 12: Facet breaking in crystalline mean curvature flow.

method is based on a parametric approach and in particular the interface is parametrized as a triangulated surface. In order to track the interface in time the triangles (or segments in 2d) move and in many numerical approaches the mesh degenerates during the evolution. In the papers [8]-[13] it was possible to come up with a variational formulation for geometrical evolution equations and free boundary problems which has the property that the mesh uses its tangential degrees of freedom in order to keep very good mesh properties. This together with a novel approach to discretize the anisotropic mean curvature H_γ made it possible to solve highly anisotropic geometric evolution equations, the Mullins-Sekerka problem and the Stefan problem with good mesh properties and a high precision. In this approach we heavily rely on an earlier work by Dziuk [52] who introduced a parametric finite element discretization of mean curvature and on work by Schmidt [146] who was the first to solve the three dimensional Stefan problem (4.24)-(4.26) with a parametric approach.

As a prototypical example for results obtained with the help of a parametric finite element method we show numerical computations of snow crystal growth, see Figures 10,11. They have been obtained with the model (4.24)-(4.26) with $\vartheta = 0$ and a hexagonal anisotropy, see Figure 8 to the bottom. Figure 10 shows the evolution of snow crystals from an initial small seed leading to forms which most of us would consider to be a typical snow crystal. In fact many other forms are also possible and I refer to Figure 11 for a form called columns on plates which also appears in nature. We refer to [11, 12] for more numerical computations and for a detailed introduction into the numerical methods. Somebody who is

interested in the fascinating aspects of the physics and mathematics of snow crystal growth can find more details in [107, 106, 77].

5 Phase field equations as gradient flows

In the geometric evolution equations and in the free boundary problems discussed so far, the interface was described as a hypersurface. In the last thirty years phase field approaches have been another successful approach to describe the evolution of interfaces. In particular phase field methods allow for a change of topology. In a phase field description of interface evolution one uses instead of a characteristic function $\chi : \Omega \to \{0, 1\}$, which describes the two regions occupied by the phases, a smooth function which takes values close to given values, e.g. ± 1, and rapidly changes between these two values in a small interfacial region, see Figures 14 and 15.

5.1 The Ginzburg–Landau energy

The phase field approach is best motivated by considering the so-called Ginzburg–Landau energy

$$E_\varepsilon(\varphi) := \int_\Omega (\tfrac{\varepsilon}{2}|\nabla\varphi|^2 + \tfrac{1}{\varepsilon}\Psi(\varphi))dx \tag{5.32}$$

where $\varepsilon > 0$ is a small parameter. For functions φ with a moderate energy $E_\varepsilon(\varphi)$ it will turn out that ε is proportional to the interfacial thickness between the region $\{\varphi \approx -1\}$ and $\{\varphi \approx 1\}$. The function $\Psi : \mathbb{R} \to \mathbb{R}_0^+$ is a double well potential having two global minima with value zero at ± 1, i.e. $\Psi(\pm 1) = 0$ and $\Psi(z) > 0$ for $z \notin \{-1, 1\}$, see Figure 13 for an example. Typical choices are the quartic potential

$$\Psi(\varphi) = \frac{9}{32}(\varphi^2 - 1)^2$$

and the double obstacle potential Ψ_{ob} which is defined as

$$\Psi_{ob}(\varphi) = \frac{1}{2}(1 - \varphi^2) \qquad \text{for all} \quad \varphi \in [-1, 1]$$

and ∞ elsewhere, see [20], although different choices are possible, see e.g. Abels and Wilke [1].

The term $\frac{1}{\varepsilon}\Psi(\varphi)$ in the energy E_ε penalizes values which differ from ± 1. In addition, the term $\frac{\varepsilon}{2}|\nabla\varphi|^2$ penalizes gradients of φ and hence too rapid changes of φ in space. It will turn out later that typical solutions of the phase field system have the form illustrated in Figure 14, i.e. they are close to ± 1 in most parts of the domain and have an interfacial region with a thickness which is proportional to ε. In directions normal to the level sets of φ a typical solution of the phase field system has the form depicted in Figure 15.

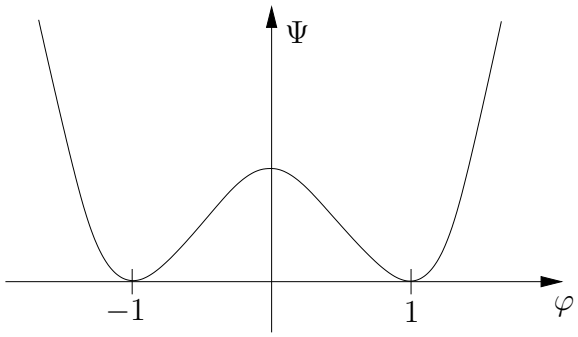


Figure 13: The energy contribution $\Psi(\varphi)$ in (5.32) penalizes values of φ, which differ from ± 1.

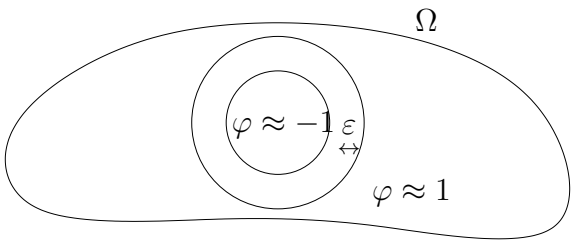


Figure 14: A typical form of the phase field variable φ. Regions in which $\varphi \approx \pm 1$, are separated by a diffuse interfacial layer whose thickness is proportional to ε.

5.2 Relating phase field and sharp interface energies

The Ginzburg-Landau energy (5.32) can be related to the surface energy in the limit $\varepsilon \to 0$. The appropriate notion to make this statement precise is the concept of Γ–limit.

DEFINITION. 5.1. *Let (X, d) be a metric space and $(F_\varepsilon)_{\varepsilon>0}$ a family of functionals $F_\varepsilon : X \to [-\infty, \infty]$. We say that $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to a functional $F : X \to [-\infty, \infty]$ (which we will denote as $F_\varepsilon \xrightarrow{\Gamma} F$) if the following properties hold:*

(i) (lim inf inequality) For every $u \in X$ and $u_\varepsilon \in X$, $\varepsilon > 0$, such that $u_\varepsilon \to u$ as $\varepsilon \to 0$ it holds

$$F(u) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon)\,.$$

(ii) (lim sup inequality) For every $u \in X$ there exist $u_\varepsilon \in X$, $\varepsilon > 0$, such that $u_\varepsilon \to u$ as $\varepsilon \to 0$ and

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \leq F(u)\,.$$

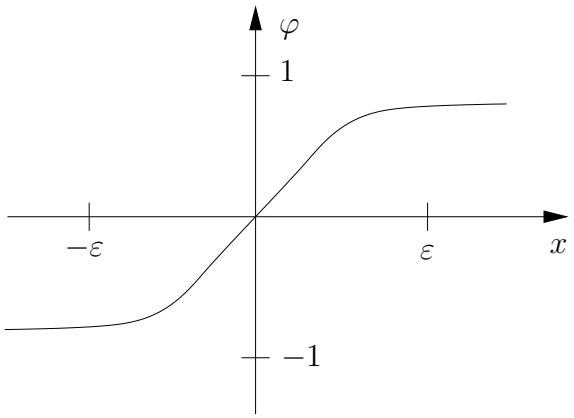


Figure 15: The phase field variable typically has a profile with a phase transition on a diffuse interface of thickness ε.

We note that the concept of Γ*-limits is more general and can be generalized to more general spaces, see [44]. The notion of* Γ*-limit is in particular appropriate for sequences of variational problems as under appropriate assumptions minima of* F_ε *will converge to minima of* F*, see [21].*

It was shown in [117] and [118] that the Ginzburg-Landau energies E_ε *defined in (5.32)* Γ*-converge to a multiple of the perimeter functional, see (2.1). It turns out that a suitable metric for this convergence is induced by the* $L^1(\Omega)$*-norm and hence we extend* E_ε *to* $L^1(\Omega)$ *by setting*

$$E_\varepsilon(\varphi) := \begin{cases} \int_\Omega (\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\Psi(\varphi))\,dx & \text{if } \varphi \in H^1(\Omega)\,, \\ \infty & \text{if } \varphi \in L^1(\Omega) \setminus H^1(\Omega)\,. \end{cases}$$

Under appropriate assumptions on Ψ *and* Ω *it can be shown that the functionals* E_ε *in fact* Γ*-converge to the functional*

$$E(\varphi) := \begin{cases} c_\Psi \int_\Omega |\nabla \chi_{\{\varphi=1\}}| & \text{if } \varphi \in BV(\Omega, \{-1,1\})\,, \\ \infty & \text{if } \varphi \in L^1(\Omega) \setminus BV(\Omega, \{-1,1\})\,, \end{cases}$$

where $c_\Psi := \int_{-1}^{1} \sqrt{2\Psi(z)}dz$*. This means we have*

$$E_\varepsilon \xrightarrow{\Gamma} E \quad \text{as} \quad \varepsilon \to 0$$

with respect to the L^1*–topology. This* Γ*-convergence result is stable under adding an integral constraint for* E_ε *in the functional which is important in many applications where this corresponds to a mass conservation property. We refer to [117, 118] and [21] for more details. In later sections we will relate gradient flows of* E_ε *to the gradient flows of the area functional* E *discussed in Section 2.*

5.3 Phase field models as gradient flows

We now consider different gradient flows involving the energy E_ε. Before discussing the gradient flows we note that the first variation $\frac{\delta E_\varepsilon}{\delta \varphi}$ of E_ε at $\varphi \in H^1(\Omega)$ in a direction $v \in H^1(\Omega)$ is given by

$$\frac{\delta E_\varepsilon}{\delta \varphi}(\varphi)(v) := \frac{d}{ds} E_\varepsilon(\varphi + sv)_{|s=0} = \int_\Omega (\varepsilon \nabla \varphi \cdot \nabla v + \tfrac{1}{\varepsilon} \Psi'(\varphi)\, v)\, dx\,.$$

5.3.1 The Allen-Cahn equation

Choosing the L^2–inner product for functions defined on Ω we now obtain the equations for the L^2–gradient flow of E_ε as follows

$$\langle \partial_t \varphi, v \rangle_{L^2} = -\int_\Omega (\varepsilon \nabla \varphi \cdot \nabla v + \tfrac{1}{\varepsilon} \Psi'(\varphi)\, v)\, dx$$

which has to hold for all times and all suitable test functions v. For functions φ which are smooth enough the above is equivalent to

$$\begin{aligned} \partial_t \varphi &= \varepsilon \Delta \varphi - \tfrac{1}{\varepsilon} \Psi'(\varphi) && \text{in } \Omega\,, \\ \frac{\partial \varphi}{\partial n} &= 0 && \text{on } \partial\Omega \end{aligned}$$

which follows after integration by parts with the help of the fundamental lemma of the calculus of variations.

5.3.2 The Cahn-Hilliard equation

It is also possible, similar as in Section 2.4 where we considered gradient flows of the area functional, to consider an H^{-1}–gradient flow of the energy E_ε which preserves the integral of φ. We define

$$H^1_m(\Omega) = \left\{ u \in H^1(\Omega) \mid \fint_\Omega u\, dx = m \right\}$$

with $m \in \mathbb{R}^+$ a given constant. For v_1, v_2 with $\int_\Omega v_i\, dx = 0$, $i = 1, 2$, we define $u_1, u_2 \in H^1_0(\Omega)$ as weak solutions of

$$\begin{aligned} -\Delta u_i &= v_i && \text{in } \Omega\,, \\ \frac{\partial u_i}{\partial n} &= 0 && \text{on } \partial\Omega\,. \end{aligned}$$

Since the u_i are the solutions of a Neumann problem for the Laplace operator we set $u_i = (-\Delta_N)^{-1} v_i$. The H^{-1}–inner product is now given as

$$\begin{aligned} \langle v_1, v_2 \rangle_{H^{-1}} &:= \int_\Omega (\nabla(-\Delta_N)^{-1} v_1) \cdot (\nabla(-\Delta_N)^{-1} v_2)\, dx \\ &= \int_\Omega \nabla u_1 \cdot \nabla u_2\, dx = \int_\Omega v_1 u_2\, dx = \int_\Omega v_2 u_1\, dx\,. \end{aligned}$$

For the H^{-1}–gradient flow we have

$$\langle \partial_t \varphi, v \rangle_{H^{-1}} = -\int_\Omega (\varepsilon \nabla \varphi \cdot \nabla v + \tfrac{1}{\varepsilon} \Psi'(\varphi) v)\, dx$$

for test functions v. Taking the definition of the H^{-1}–inner product into account we observe after integration by parts that a smooth solution of the gradient flow equation is a solution of the following boundary value problem:

$$\partial_t \varphi = \Delta(-\varepsilon \Delta \varphi + \tfrac{1}{\varepsilon} \Psi'(\varphi)) \quad \text{in } \Omega\,, \tag{5.33}$$

$$\tfrac{\partial \varphi}{\partial n} = 0\,, \qquad \tfrac{\partial \Delta \varphi}{\partial n} = 0 \qquad \text{on } \partial\Omega\,. \tag{5.34}$$

The facts that the Allen-Cahn equation and the Cahn-Hilliard equation are respectively the L^2– and the H^{-1}–gradient flow of the Ginzburg-Landau energy E_ε has first been discussed by Fife [70, 71]. The equation (5.33) is a parabolic partial differential equation of fourth order which is called the Cahn-Hilliard equation, see [57] and [129] for more details. Solutions of (5.33), (5.34) fulfill

$$\frac{d}{dt} \int_\Omega \varphi \, dx = 0\,, \qquad \frac{d}{dt} E_\varepsilon(\varphi) \le 0$$

which are the analogues of Proposition 2.1 which stated the related result for surface diffusion which is the H^{-1}–gradient flow of the area functional.

For later use we remark that a variant of the Cahn-Hilliard equation has a degenerate mobility $M(\varphi) := (1-\varphi^2)_+ := \max(0, 1-\varphi^2)$ and in this case we replace (5.33) by

$$\partial_t \varphi = \nabla \cdot (M(\varphi) \nabla(-\varepsilon \Delta \varphi + \tfrac{1}{\varepsilon} \Psi'(\varphi)))\,.$$

How to obtain this equation as a gradient flow of a suitably weighted H^{-1}–inner product is discussed by Taylor, Cahn [158] and an existence analysis for this equation is given in [58].

5.3.3 The phase field system

It is also possible to formulate a phase field analogue of the full Stefan problem (4.24)-(4.26). We derive a simplified version of the phase field system, similar as in a paper by Penrose and Fife [133] with the help of the gradient flow perspective. To this end we consider the unknowns interfacial energy e and phase field φ for which we define the functional

$$E(e, \varphi) = \int_\Omega (s(e, \varphi) + \tfrac{\varepsilon}{2} |\nabla \varphi|^2 + \tfrac{1}{\varepsilon} \Psi(\varphi)) dx\,.$$

We now take the inner product $\langle e_1, e_2 \rangle_{H^{-1}} + \langle \varphi_1, \varphi_2 \rangle_{L^2}$ and obtain as gradient flow (not taking boundary conditions into account)

$$(-\Delta_N)^{-1} \partial_t e = -\frac{\delta E}{\delta e}\,, \tag{5.35}$$

$$\partial_t \varphi = -\frac{\delta E}{\delta \varphi}\,. \tag{5.36}$$

Defining $s(e,\varphi) = \frac{1}{2}(e-\varphi)^2$ and $u = e - \varphi$ we obtain

$$\frac{\partial s}{\partial e} = u \quad , \quad \frac{\partial s}{\partial \varphi} = -u$$

and hence we can rewrite (5.35), (5.36) as

$$\begin{aligned} \partial_t(u+\varphi) &= \Delta u\,, & (5.37)\\ \partial_t \varphi &= \varepsilon\Delta\varphi - \tfrac{1}{\varepsilon}\Psi'(\varphi) + u\,. & (5.38) \end{aligned}$$

This is the phase field system and u typically is interpreted as temperature or chemical potential. We refer to [26, 33, 78, 133, 153, 165] for more information on the phase field model which in particular discuss thermodynamically consistent phase field models.

5.4 Sharp interface limits

It is possible to relate the gradient flows for the area functional and the gradient flows for the Ginzburg-Landau functional. In all situations discussed above the time dependent solutions $(\varphi_\varepsilon)_{\varepsilon>0}$ of a phase-field type equation converge to a function φ_0 taking only values ± 1. The boundary between the sets $\{\varphi_0 = 1\}$ and $\{\varphi_0 = -1\}$ is a (maybe "generalized") surface for which an evolution law similar as the ones discussed in Sections 2 and 3 will hold.

1.) *The Allen-Cahn equation*
After a suitable rescaling in time the Allen-Cahn equation is given as

$$\varepsilon\partial_t\varphi_\varepsilon - \varepsilon\Delta\varphi_\varepsilon + \tfrac{1}{\varepsilon}\Psi'(\varphi_\varepsilon) = 0\,.$$

In the limit $\varepsilon \to 0$ we obtain that the surface separating the sets $\{\varphi_0 = 1\}$ and $\{\varphi_0 = -1\}$, compare the Figures 14 and 15, will evolve by mean curvature flow

$$V = H\,.$$

There are many results on this limit using quite different methods. We refer e.g. to [24, 34, 48, 68, 102] for details.

2.) *The Cahn-Hilliard equation*
In order to discuss the sharp interface limit of the Cahn-Hilliard equation (5.33) we restate the equation as a system

$$\begin{aligned} \partial_t\varphi_\varepsilon &= \Delta u_\varepsilon\,, & (5.39)\\ u_\varepsilon &= -\varepsilon\Delta\varphi_\varepsilon + \tfrac{1}{\varepsilon}\Psi'(\varphi_\varepsilon)\,. & (5.40) \end{aligned}$$

In the limit $\varepsilon \to 0$ one obtains the Mullins-Sekerka problem, see (2.15)-(2.17),

$$\begin{aligned} 0 = &\quad \Delta u && \text{in } \Omega_- \cup \Omega_+ \,, && (5.41)\\ 2V = &\quad -[\nabla u]^+_- \cdot \nu && \text{on } \Gamma \,, && (5.42)\\ 2u = &\quad c_\Psi H && \text{on } \Gamma \,. && (5.43) \end{aligned}$$

We obtain the factor 2 in (5.42) in comparison to (2.16) because φ jumps from -1 to 1 whereas in (2.16) we considered the characteristic function of phase 2 which has the jump one across the interface. The different factor on the right hand side in (5.43) in comparison to (2.17) is explained by the fact that in E the surface area is weighted by the factor c_Ψ. The asymptotic limit of the Cahn-Hilliard equation has been studied in [4, 132, 154].

3.) *The nonlocal Allen-Cahn equation*
For the area functional we considered an L^2–gradient flow which preserves the enclosed volume by requiring that the normal velocities have mean zero. One can do something similar for the Ginzburg-Landau energy and obtains a gradient flow which, taking Neumann boundary conditions into account, preserves the integral of φ:

$$\varepsilon \partial_t \varphi_\varepsilon - \varepsilon \Delta \varphi_\varepsilon + \tfrac{1}{\varepsilon} \Psi'(\varphi_\varepsilon) = \tfrac{1}{\varepsilon} \fint_\Omega \Psi'(\varphi_\varepsilon) \, dx \,.$$

As asymptotic limit for $\varepsilon \to 0$ one obtains the nonlocal mean curvature flow, compare (2.8),

$$V = H - \fint_\Gamma H \,.$$

This asymptotic limit was studied by [25] in the radially symmetric case and in [36] for general geometries.

4.) *The Cahn-Hilliard equation with degenerate mobility*
In Subsection 2.4 we studied two different H^{-1}-gradient flows of the area functional. One was motion by surface diffusion, a local geometric evolution law, and the second was the Mullins-Sekerka free boundary problem. It turns out that we obtain both as asymptotic limits of Cahn-Hilliard equations. We already saw that the Mullins-Sekerka evolution is the sharp interface limit of the Cahn-Hilliard equation.

Taking a degenerate mobility in the Cahn-Hilliard equation and rescaling in time we have

$$\begin{aligned} \varepsilon \partial_t \varphi_\varepsilon &= \nabla \cdot ((1 - \varphi_\varepsilon^2)_+ \nabla u_\varepsilon) \,, \\ u_\varepsilon &= -\varepsilon \Delta \varphi_\varepsilon + \tfrac{1}{\varepsilon} \Psi'(\varphi_\varepsilon) \end{aligned}$$

and formal arguments by Cahn, Elliott and Novick-Cohen [30] indicate that solutions of this system converge to motion by surface diffusion, i.e. the limiting evolving surface fulfills

$$V = -d_\Psi \Delta_\Gamma H \text{ with a suitable constant } d_\Psi \in \mathbb{R}\,.$$

We remark here that it is still an open problem to justify this limit rigorously.

5.) *The phase field system*
It is also possible to relate the phase field equations to the Stefan problem (4.24)-(4.26) discussed in Section 4. We restrict ourselves to the isotropic case, i.e. in (4.24)-(4.26) we choose β as constant and $H_\gamma = \gamma H$ with a constant γ. In order to obtain the isotropic variant of (4.24)-(4.26) we formulate (5.37)-(5.38) with the help of physical constants as follows

$$\begin{aligned} \partial_t(\vartheta u + \tfrac{\lambda}{2}\varphi) &= \mathcal{K}\Delta u\,, && (5.44)\\ c_\Psi \tfrac{a}{2} u &= \varepsilon\beta\partial_t\varphi - \gamma\varepsilon\Delta\varphi + \tfrac{\gamma}{\varepsilon}\Psi'(\varphi)\,. && (5.45) \end{aligned}$$

It can be shown that this system converges to (4.24)-(4.26) (for $\mathcal{K}_- = \mathcal{K}_+$) in the sharp interface limit $\varepsilon \to 0$. This has been analyzed with the help of formally matched asymptotic expansions by Caginalp [27, 28], see also [53], and was later shown rigorously by Caginalp and Chen [29]. We remark that (5.44), (5.45) also contains the Allen-Cahn equation and the Cahn-Hilliard equation as special cases by setting either $a = 0$ or ($\vartheta = 0$, $\beta = 0$). Also the case $\vartheta = 0$ is of importance and is called the viscous Cahn-Hilliard equation, see [128]. By choosing $\beta = 0$ one obtains the Stefan problem with Gibbs-Thomson law as the asymptotic limit, see [134, 144].

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