The dynamical behaviour of the support splitting and merging phenomena appearing in the flow through an absorbing medium

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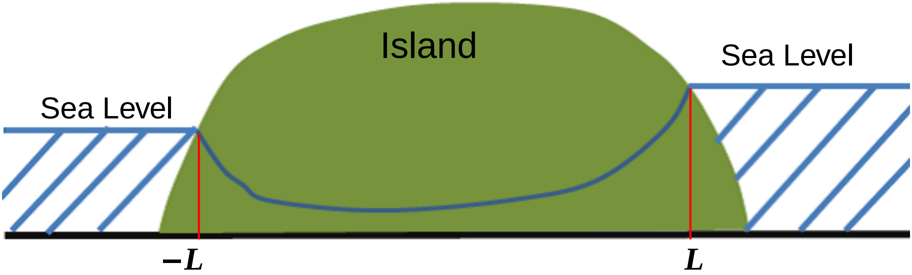
1. Introduction.

Numerical experiments suggest the interesting properties in the several fields.

Let us consider the flow through an absorbing medium with

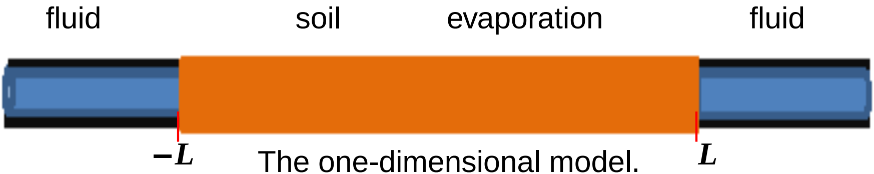
"penetration of the fluid from the boundary",

on which the flowing tide and the ebbing tide occur.



The distribution of the density of the fluid in $[-L, L]$

In this talk we treat the simplest case. Assume the density of the fluid is uniform along the vertical line.



The one-dimensional model.

Then the interaction between diffusion and absorption may cause dynamical behaviour of support,

- “support splitting phenomena”,
- “support splitting and merging phenomena”,
- “repeated support splitting and merging phenomena”.

Here, support means the region occupied by the fluid.

To realize such phenomena in $\mathbf{R}^1$ we consider the following Model equation = Nonlinear diffusion + Absorption, which is written in the form of the initial-boundary value problem:

$$\text{(IBVP)} \quad \begin{cases} v_t(t,x) = (v^m)_{xx} - cv^p \ \text{ on } \ (0,\infty) \times (-L,L), \\ v(t,-L) = f(t), \quad v(t,L) = g(t) \ \text{ for } \ t \geq 0, \\ v(0,x) = v^0(x) \ \text{ on } \ [-L,L], \end{cases}$$

where v denotes the density of the liquids,
$m > 1,\ 0 < p < 1,\ c > 0,\ m + p \geq 2$, and $v^0(x)(\geq 0)$.

- The existence and uniqueness of a weak solution is established by Oleinik, Kalashnikov and Chzou(1958), Kalashnikov(1974,1987), Kersner(1978,1980) and Bertsch(1980).

- Kersner(1980) proved the support splitting phenomena by constructing the supersolution.

For Problem (IBVP) we consider the following two cases.

Case A) $f(t)$, $g(t) \equiv Const.$, i.e, independent of t.

Case B) $f(t)$, $g(t)$ are periodic functions with respect to t.

Our aims:

- In Case A) investigate the profiles of the stationary solutions satisfying

$$\text{(BVP)} \begin{cases} (v^m)_{xx} - cv^p = 0 \ \text{ on } \ (-L, L), \\ v(-L) = \alpha, \quad v(L) = \beta. \end{cases}$$

- In Case B) find interesting support splitting and merging phenomena.
  Here, the flowing tide and the ebbing tide causes the penetration of the fluid from the boundary.

My plan is as follows:

2. Numerical schemes.

3. Numerical experiments in Case A).

Stationary solutions and stabilization.

Split in finite time.

4. Numerical experiments in Case B).

repeated support splitting and merging phenomena.

5. Conclusions.

2. Numerical schemes.

$$(*) \quad v_t(t,x) = (v^m)_{xx} - cv^p \text{ on } (0,\infty)\times(-L,L).$$

Putting $u \equiv v^{m-1}$, we rewrite the original equation $(*)$ as

$$(**) \quad u_t = muu_{xx} + a(u_x)^2 - (m-1)cu^q, \; q = \frac{m+p-2}{m-1}.$$

Our scheme approximates $(**)$ instead of the original $(*)$.

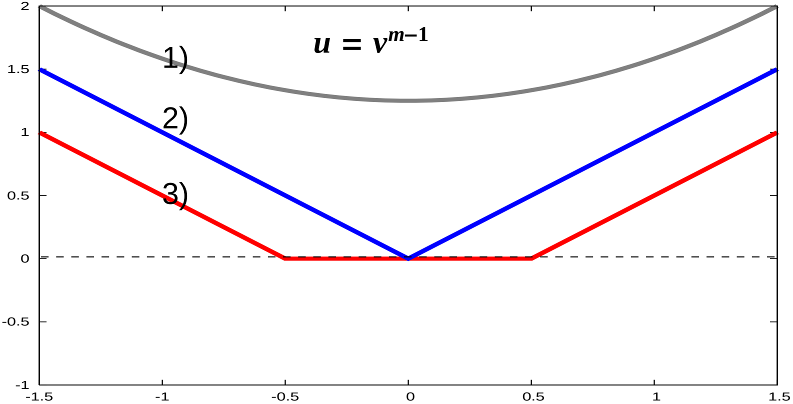
In the following we show numerical solutions in the case where

$$m + p = 2.$$

3. Numerical experiments in Case A).

Let $m = 1.5, p = 0.5, c = 6$.

Numerical solutions u_h^n to $u \equiv v^{m-1}$.



1) $u(0, x) = 2.0, \quad u(t, \pm 1.5) = 2.0 \rightarrow$ no-split (file=N9528-0ReduceDATA02.swf)

2) $u(0, x) = 1.5, \quad u(t, \pm 1.5) = 1.5 \rightarrow$ no-split (file=N9528-1ReduceDATA02.swf)

3) $u(0, x) = 1.0, \quad u(t, \pm 1.5) = 1.0 \rightarrow$ split (file=N9528-2ReduceDATA02.swf)

Stationary solutions and stabilization.

$$\text{(IBVP)}\begin{cases} v_t(t,x) = (v^m)_{xx} - cv^p \text{ on } (0,\infty)\times(-L,L),\ (m+p\geq 2), \\ v(t,-L)=\alpha, \quad v(t,L)=\beta \text{ for } t\geq 0, \\ v(0,x)=v^0(x) \text{ on } [-L,L], \end{cases}$$

$$\text{(BVP)}\begin{cases} (v^m)_{xx} - cv^p = 0 \text{ on } (-L,L), \\ v(-L)=\alpha, \quad v(L)=\beta. \end{cases}$$

Stabilization theorem. In (IBVP) suppose that $v^0(x)$ is continuous. Then the stationary solution $\tilde{v}$ (sol. of (BVP)) exists and is unique, and

$$v(t,\cdot) \longrightarrow \tilde{v} \text{ in } C[-L',\ L'] \text{ as } t\to\infty,$$

where $[-L',\ L'] \subset (-L,\ L)$ is an arbitrary fixed interval.

Moreover, stationary solutions are classified into three types.

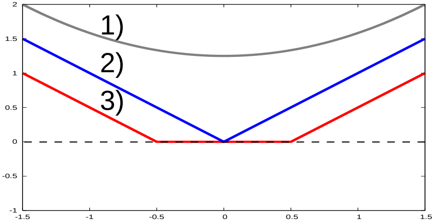
Split in finite time.

In Stabilization theorem, we have

$v(t,\cdot) \longrightarrow$ 1) – No Split

$v(t,\cdot) \longrightarrow$ 2) – No Split

$v(t,\cdot) \longrightarrow$ 3) – Split



Theorem. Let $m + p = 2$.

When $v(t,\cdot) \longrightarrow$ 3), the support splits in finite time; that is, there exist some T^* and interval $[-\ell', \ell'] \subset (-L', L')$ such that

$$v(t,x) = 0 \text{ on } [-\ell', \ell'] \text{ for all } t > T^*.$$

4. Numerical experiments in Case B).

$$\text{(IBVP)}\begin{cases} v_t(t,x) = (v^m)_{xx} - cv^p \text{ on } (0,\infty)\times(-L,L), \ (m+p\geq 2), \\ v(t,-L) = f(t), \quad v(t,L) = g(t) \text{ for } t \geq 0, \\ v(0,x) = v^0(x) \text{ on } [-L,L], \end{cases}$$

Let $f(t) \equiv g(t)$ be a periodic function.

We show numerical experiments in case where

$$m + p = 2.$$

Let $m = 1.5, p = 0.5, c = 6$, $u(t,x) = v(t,x)^{m-1}$.

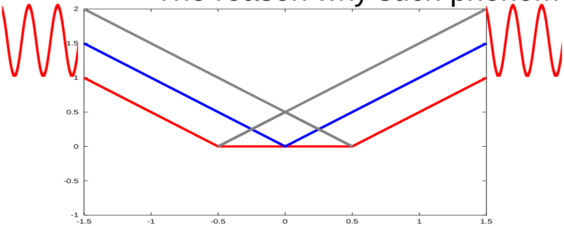
4) $u(0,x) = 2.0,\ u(t,\pm L) \equiv \varphi(t) = 1.5 + 0.5\cos(2\pi t)$

→ split-merge (N9528-3ReduceDATA02.swf)

5) $u(0,x) = 1.0,\ u(t,\pm L) \equiv \varphi(t) = 1.5 - 0.5\cos(16\pi t)$

→ no-split (file=N9528-3FReduceDATA02.swf)

- - - The reason why such phenomena appear ?



Taking account of the stabilization theorem, we have

Theorem. Let $m + p = 2$.
When $\varphi(t)$ takes a sufficiently long period in 4), there appear repeated support splitting and merging phenomena.

5. Conclusions.

$$\text{(IBVP)} \begin{cases} v_t(t,x) = (v^m)_{xx} - cv^p \text{ on } (0,\infty)\times(-L,L), \\ v(t,-L) = f(t), \quad v(t,L) = g(t) \text{ for } t \geq 0, \\ v(0,x) = v^0(x) \text{ on } [-L,L], \end{cases}$$

$$m > 1,\ 0 < p < 1,\ c > 0,\ m + p \geq 2.$$

1) Stabilization Theorem, that is, when $f(t), g(t) = Const.$,
   $v(t,x) \to$ "stationary solution"$\tilde{v}(x)$ as $t \to \infty$.

2) Repeated support splitting and merging phenomena
   appear in $m + p = 2.$

3) Repeated support splitting and merging phenomena
   appear in $m + p \neq 2$?

4) The relation between
   support non-splitting phenomena and the period of $\varphi(t)$?

Thank you very much for your attention !