Evolution of viscous conducting drops subject to rotation and electric fields

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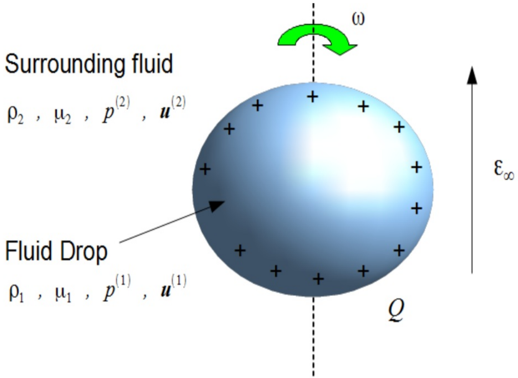
Mathematical Model



Two cases: $L = \mathcal{I}\omega$

- Electric field $\mathcal{E}_\infty$ and constant L.
- Charge Q and constant L.

Notation

$\mathcal{D}_1(t)$ Fluid drop domain
ρ_1 Drop density
μ_1 Drop viscosity
$p^{(1)}$ Drop pressure
$\mathbf{u}^{(1)}$ Drop velocity

$\mathcal{D}_2(t)$ Outer fluid domain
ρ_2 Outer fluid density
μ_2 Outer fluid viscosity
$p^{(2)}$ Outer fluid pressure
$\mathbf{u}^{(2)}$ Outer fluid velocity

$\boldsymbol{\omega}$ Angular velocity
$\partial\mathcal{D}(t)$ Fluid interface

Mathematical Model

Stokes equation

$$\begin{cases} \mu_i \Delta \mathbf{u}^{(i)} - \nabla \Pi^{(i)} = \mathbf{0} & , \quad \text{in } \mathcal{D}_i\left(t\right) \\ \nabla \cdot \mathbf{u}^{(i)} = 0 & , \quad \text{in } \mathcal{D}_i\left(t\right) \end{cases} \quad , \quad i \in \{1, 2\}$$

Boundary condition

$$\left(T^{(2)} - T^{(1)}\right) \mathbf{n} = \left(2\gamma\mathcal{H} - \varrho \frac{L^2}{2\mathcal{I}^2} r_{axis}^2 - \frac{\sigma^2}{2\varepsilon_0}\right) \mathbf{n} \quad , \quad \text{on } \partial\mathcal{D}\left(t\right)$$

Laplace equation

$$\begin{cases} \Delta \mathcal{V} = 0 & , \quad \text{in } \mathcal{D}_2\left(t\right) \\ \mathcal{V} = \mathcal{V}_0 & , \quad \text{in } \partial\mathcal{D}\left(t\right) \\ \mathcal{V} \to -\mathcal{E}_\infty z + O(|\mathbf{r}|^{-1}) & , \quad \text{as } |\mathbf{r}| \to \infty \end{cases}$$

Boundary integral formulation

Fredholm integral equation of the 2^{nd} kind for the velocity:

$$u_j(\mathbf{r}') = -\frac{1}{4\pi(\mu_1+\mu_2)} \underbrace{\int_{\partial\mathcal{D}(t)} f_i(\mathbf{r}) G_{ij}(\mathbf{r},\mathbf{r}')\, dS(\mathbf{r})}_{\text{Single layer potential}}$$

$$- \frac{\mu_1-\mu_2}{4\pi(\mu_1+\mu_2)} \underbrace{\int_{\partial\mathcal{D}(t)}^{PV} u_i(\mathbf{r}) T_{ijk}(\mathbf{r},\mathbf{r}') n_k(\mathbf{r})\, dS(\mathbf{r})}_{\text{Double layer potential}}$$

$$G_{ij}(\mathbf{r},\mathbf{r}') = \frac{\delta_{ij}}{|\mathbf{r}-\mathbf{r}'|} + \frac{(r_i-r_i')(r_j-r_j')}{|\mathbf{r}-\mathbf{r}'|^3} \qquad \text{Stokeslet}$$

$$T_{ijk}(\mathbf{r},\mathbf{r}') = -6\frac{(r_i-r_i')(r_j-r_j')(r_k-r_k')}{|\mathbf{r}-\mathbf{r}'|^5} \qquad \text{Tenselet}$$

$$f_i(\mathbf{r}) = \left[2\gamma\mathcal{H}(\mathbf{r}) - \varrho\frac{L^2}{2\mathcal{I}^2} r_{axis}^2 - \frac{\sigma^2}{2\varepsilon_0}\right] n_i(\mathbf{r}) \qquad \text{Traction}$$

where $i,j,k \in \{1,2,3\}$ and $\mathbf{r}$, $\mathbf{r}' \in \partial\mathcal{D}(t)$.

Numerical algorithm

Algorithm

1. Compute the volume and moment of inertia about the z-axis.
2. Calculate mean curvature and charge density of drop.
3. Solve linear system to obtain velocity field at the boundary.
4. Move the boundary with an Euler explicit scheme:

$$\mathbf{r}\left(t_{n+1}\right) = \mathbf{r}\left(t_n\right) + \mathbf{u}\left(t_n\right)\Delta t \ .$$

5. Regularization of the mesh (if necessary):
   - Delaunay remeshing.
   - Mesh relaxation.
   - Mesh refinement.

Repeat above steps until $t_{\max}$ is reached.

Charge and rotation at constant L

Asymptotic expansion: $L^2 \ll 1$

Young-Laplace equation

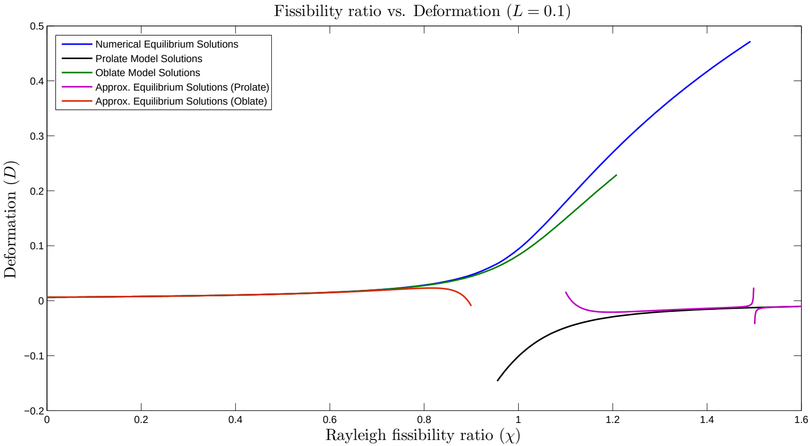
$$\delta p = 2\gamma\mathcal{H} - \varrho\frac{L^2}{2\mathcal{I}^2}r^2_{axis} - \frac{\sigma^2}{2\varepsilon_0} \quad , \quad \text{on } \partial\mathcal{D}$$

Spheroidal approximation:

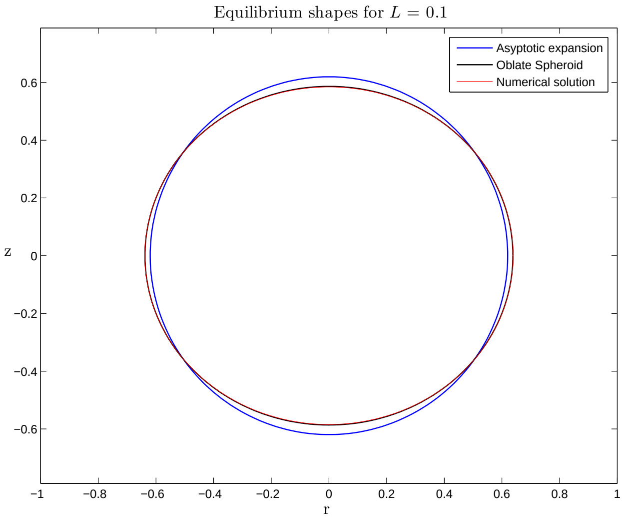
Energy formulation

Minimize: $E_{total} = E_{area} + E_{kinetic} + E_{electrostatic} \quad , \quad V = 1$

Charge and rotation at constant L



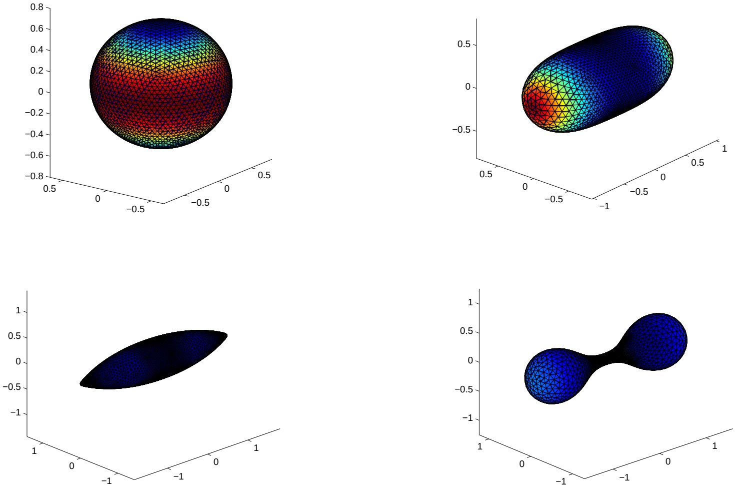
Charge and rotation at constant L



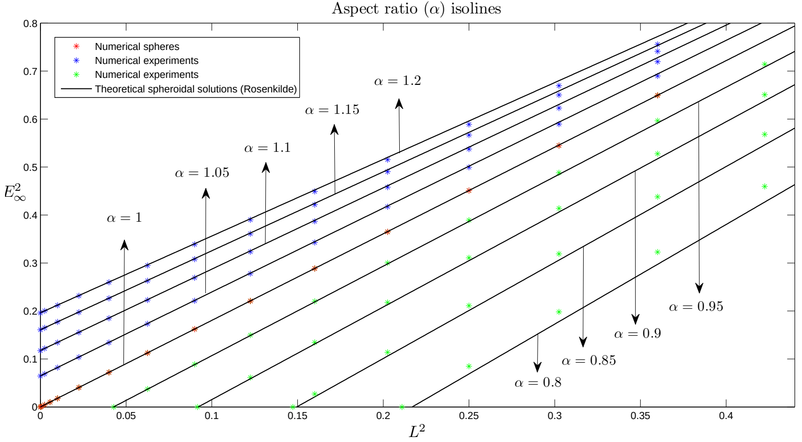
$$\chi = 0.892$$

Charge and rotation at constant L

Stability analysis

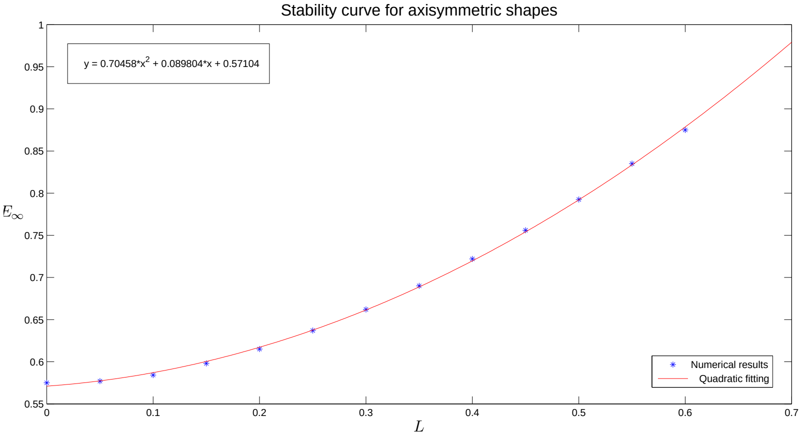


Electric field and rotation at constant L

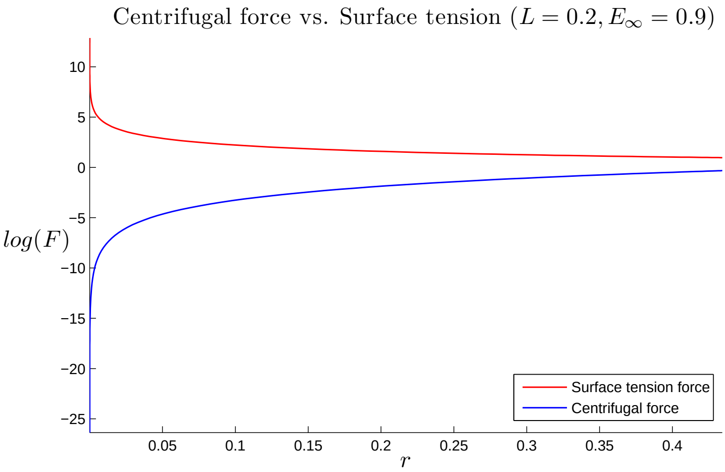


$$E_\infty^2 = h\left(\alpha\right) + g\left(\alpha\right) L^2$$

Electric field and rotation at constant L

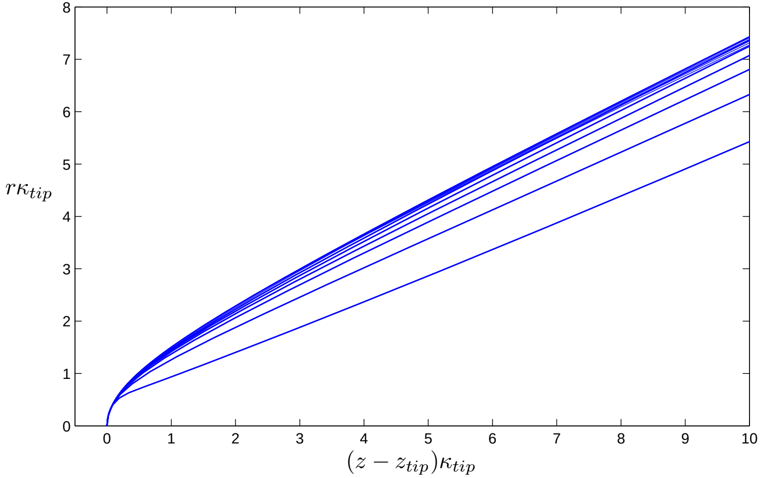


Taylor cones

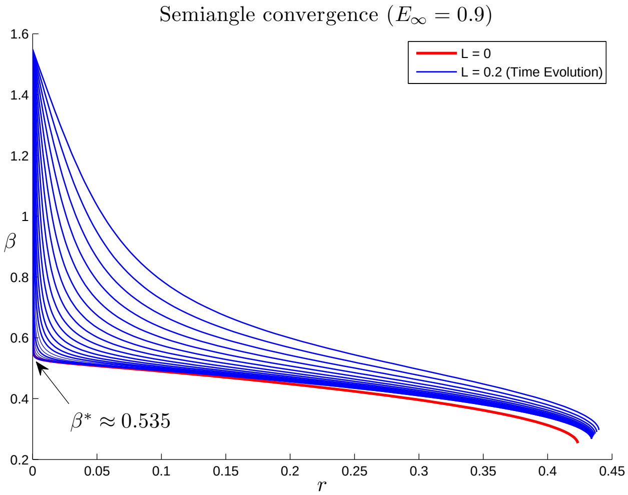


Taylor cones

Self-similarity



Taylor cones



Conclusions & Future Research Lines

Conclusions

- Adaptive BEM to simulate droplet evolution.
- Theoretical models to approximate charged rotating drops.
- Stability analysis shows ellipsoidal configurations and singularities (Taylor cones and two-lobed drop breakup).
- Linear relationship for shapes of rotating drops subject to uniform electric fields with same aspect ratio.
- Taylor cone semiangle is not affected by small rotations.

Research Lines

- Evolution of charged rotating drops subject to electric fields.
- Describe the space of parameters (χ, E_∞, L).
- Understand the role of rotation on the stability of the system.

Thank you for your attention.

Questions?

Mean curvature

Algorithm 1 Paraboloid fitting

1: Take $\mathbf{n}_p$ initial approximation to the outward unit normal to $\partial\mathcal{D}$ at $\mathbf{p}$.
2: **repeat**
3: Choose local coordinates (x', y', z') at $\mathbf{p}$ and z'-axis along $\mathbf{n}_p$.
4: Find (x'_i, y'_i, z'_i) coordinates of the adjacent nodes to $\mathbf{p}$.
5: Minimize:

$$F = \sum_{i=1}^{N_p} \left(Ax'_i + By'_i + C(x'_i)^2 + Dx'_i y'_i + E(y'_i)^2 - z'_i \right)^2 \ .$$

6: $(\mathbf{n}_p)_n \leftarrow (-A, -B, 1)/(1 + A + B)^{1/2}$.
7: **until** $|(\mathbf{n}_p)_n - \mathbf{n}_p| < \varepsilon$.
8: Mean curvature $k_{\mathbf{p}} = \dfrac{(1+B^2)C - ABD + (1+A^2)E}{(1+A^2+B^2)^{3/2}}$.

Fast convergence iterative method

Surface charge density

Write:

$$\sigma = \mathcal{V}_0 \sigma_0 + \sigma_{ind}$$

Solve:

$$\mathcal{E}_\infty z_0 = \frac{1}{4\pi\varepsilon_0} \int_{\partial\mathcal{D}(t)} \frac{\sigma_{ind}(\mathbf{x})}{\|\mathbf{x}-\mathbf{x}_0\|} \, dS \quad , \quad 1 = \frac{1}{4\pi\varepsilon_0} \int_{\partial\mathcal{D}(t)} \frac{\sigma_0(\mathbf{x})}{\|\mathbf{x}-\mathbf{x}_0\|} \, dS$$

with:

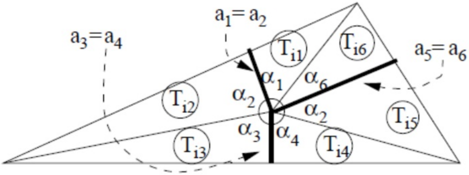
$$Q = \mathcal{V}_0 \int_{\partial\mathcal{D}(t)} \sigma_0(\mathbf{x}) \, dS + \int_{\partial\mathcal{D}(t)} \sigma_{ind}(\mathbf{x}) \, dS$$

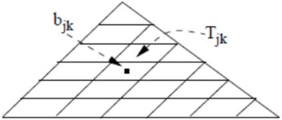
Linear system at barycenters $\mathbf{x}_l$, $\mathbf{x}_j$ of triangles:

$$\int_{\partial\mathcal{D}(t)} \frac{\sigma(\mathbf{x})}{\|\mathbf{x}-\mathbf{x}_i\|} \, dS \approx \sum_{j=1}^{M} \lambda_{ij}\sigma_j \quad , \quad \lambda_{ij} = \int_{T_j} \frac{dS}{\|\mathbf{x}-\mathbf{x}_i\|}$$

$$\sigma_j = \sigma(\mathbf{x}_j)$$

Surface charge density





- Case $i = j$

$$\lambda_{ii} = \sum_{k=1}^{6} \int\int_{T_{ik}} d\rho \, d\theta = \sum_{k=1}^{6} a_k \ln\left(\sec\left(\alpha_k\right) + \tan\left(\alpha_k\right)\right)$$

- Case $i \neq j$

$$\lambda ij = \sum_{k=1}^{N_s} \lambda_{ij,k} \quad , \quad \lambda_{ij,k} = \frac{Area\left(T_{jk}\right)}{\left\|\mathbf{b}_{jk} - \mathbf{x}_i\right\|}$$