On a free boundary problem of magnetohydrodynamics in multi-connected domains

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1 Formulation of the problem

Domains

Ω_{1t}: variable domain filled with a liquid,
Ω_{2t}: vacuum region surrounding Ω_{1t},
Ω_3: fixed domain where the electric current $\boldsymbol{j}(x,t)$ is given;
$\Omega = \overline{\Omega}_{1t} \cup \overline{\Omega}_3 \cup \Omega_{2t}$,
$\Gamma_t = \partial\Omega_{1t}$: free surface, S_3: the boundary of Ω_3,
S: the boundary of Ω–a perfect conductor.

Equations of magnetohydrodynamics.

Navier-Stokes equations:

$$\begin{cases} \boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nabla\cdot(T(\boldsymbol{v},p) + T_M(\boldsymbol{H})) = \boldsymbol{f}(x,t), \\ \nabla\cdot\boldsymbol{v}(x,t) = 0, \quad x\in\Omega_{1t}, \ \ t>0, \end{cases} \tag{1.1}$$

Maxwell equations:

$$\begin{cases} \mu\boldsymbol{H}_t = -rot\boldsymbol{E}, \quad \nabla\cdot\boldsymbol{H} = 0, \quad x\in\Omega_{1t}\cup\Omega_{2t}\cup\Omega_3, \\ rot\boldsymbol{H} = \alpha(\boldsymbol{E} + \mu(\boldsymbol{v}\times\boldsymbol{H})), \quad x\in\Omega_{1t}, \\ rot\boldsymbol{H} = 0, \quad \nabla\cdot\boldsymbol{H} = 0, \quad \nabla\cdot\boldsymbol{E} = 0, \quad x\in\Omega_{2t}, \\ rot\boldsymbol{H} = \alpha\boldsymbol{E} + \boldsymbol{j}(x,t), \quad x\in\Omega_3. \end{cases} \tag{1.2}$$

$\boldsymbol{v}, p$: velocity and pressure, $\boldsymbol{f}$: mass forces,
$\boldsymbol{H}(x,t)$, $\boldsymbol{E}(x,t)$: magnetic and electric fields,
μ: piecewise constant positive function equal to μ_i in Ω_i
α: piecewise constant function positive in Ω_{1t}, Ω_3 and $\alpha = 0$ in Ω_{2t},
$T(\boldsymbol{v},p) = -pI + \nu S(\boldsymbol{v})$: the viscous stress tensor,
$S(\boldsymbol{v}) = \nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T$: the doubled rate-of-strain tensor,
$T_M(\boldsymbol{H}) = \mu(\boldsymbol{H}\otimes\boldsymbol{H} - \frac{1}{2}|\boldsymbol{H}|^2 I)$: the magnetic stress tensor.

Boundary and jump conditions:

$$\begin{cases} \boldsymbol{H}\cdot\boldsymbol{n}=0, \quad \boldsymbol{E}_\tau=0, \quad x\in S, \\ [\mu\boldsymbol{H}\cdot\boldsymbol{n}]=0, \quad [\boldsymbol{H}_\tau]=0, \quad [\boldsymbol{E}_\tau]=0, \quad x\in S_3, \\ \big(T(\boldsymbol{v},p)+[T_M(\boldsymbol{H})]\big)\boldsymbol{n}=0, \quad V_n=\boldsymbol{v}\cdot\boldsymbol{n}, \\ [\mu\boldsymbol{H}\cdot\boldsymbol{n}]=0, \quad [\boldsymbol{H}_\tau]=0, \\ \mathfrak{n}_t[\mu]\boldsymbol{H}_\tau+[\mathfrak{n}_x\times\boldsymbol{E}]=0, \quad x\in\Gamma_t, \end{cases} \tag{1.3}$$

$\boldsymbol{H}_\tau=\boldsymbol{H}-\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{H})$, $\boldsymbol{E}_\tau=\boldsymbol{E}-\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{E})$: tangential components of $\boldsymbol{H}$ and $\boldsymbol{E}$,
$[\boldsymbol{F}]$: jump of the vector field $\boldsymbol{F}(x)$ on Γ_t and S_3,
V_n: the velocity of evolution of Γ_t in the direction of the exterior normal $\boldsymbol{n}$,
$\mathfrak{n}=(\mathfrak{n}_x,\mathfrak{n}_t)$: normal to the surface $x\in\Gamma_t$, $t\in(0,T)$ in $\mathbb{R}^4$.

Initial and orthogonality conditions

$$\begin{aligned} &\boldsymbol{v}(x,0)=\boldsymbol{v}_0(x), \quad x\in\Omega_1, \\ &\boldsymbol{H}(x,0)=\boldsymbol{H}_0(x), \quad x\in\Omega_1\cup\Omega_2\cup\Omega_3, \end{aligned} \tag{1.4}$$

$$\int_{S_k}\boldsymbol{E}\cdot\boldsymbol{n}dS=0. \tag{1.5}$$

S_k: connected components of the boundary of Ω_{2t}, $\Omega_i=\Omega_{i0}$, $i=1,2,3$

Compatibility conditions

$$\begin{aligned} &\nabla\cdot\boldsymbol{v}_0(x)=0, \quad x\in\Omega_1, \\ &\nabla\cdot\boldsymbol{H}_0(x)=0, \quad x\in\Omega_1\cup\Omega_2\cup\Omega_3, \\ &rot\boldsymbol{H}_0(x)=0, \quad x\in\Omega_2, \\ &[(S(\boldsymbol{v}_0)\boldsymbol{n})_\tau]=0,\ [\boldsymbol{H}_{0\tau}]=0,\ [\mu\boldsymbol{H}_0\cdot\boldsymbol{N}]=0, \quad x\in\Gamma_0, \\ &[\boldsymbol{H}_{0\tau}]=0,\ [\mu\boldsymbol{H}_0\cdot\boldsymbol{n}]=0,\ x\in S_3, \\ &\boldsymbol{H}_0(x)\cdot\boldsymbol{n}=0, \quad x\in S. \end{aligned} \tag{1.6}$$

2 Transformation of the problem and formulation of main result.

Lagrangian coordinates $\xi\in\Omega_1$:

$$x=\xi+\int_0^t\boldsymbol{u}(\xi,\tau)d\tau\equiv X(\xi,t), \quad \boldsymbol{u}(\xi,t)=\boldsymbol{v}(X(\xi,t),t),$$
$$X:\Omega_1\to\Omega_{1t}, \quad \Gamma_0\to\Gamma_t.$$

$\boldsymbol{u}^*(x,t)$: the extension of $\boldsymbol{u}$ from Ω_1 in Ω such that $\boldsymbol{u}^*(\xi,t)=0$ in Ω_3 and near $S_3\cup S$.

$$x=\xi+\int_0^t\boldsymbol{u}^*(\xi,\tau)d\tau\equiv X(\xi,t),$$
$$X:\Omega_i\to\Omega_{it}, \quad i=1,2, \quad \Omega_3\to\Omega_3,$$
$$\mathcal{L}(\boldsymbol{u})\equiv\Big(\frac{\partial x}{\partial\xi}\Big), \quad L(\boldsymbol{u})=det\mathcal{L}, \quad \widehat{\mathcal{L}}=L\mathcal{L}^{-1};$$

$L=1$ for $\xi \in \Omega_1$, $\widehat{\mathcal{L}}^T \equiv A(\boldsymbol{u})$: co-factors matrix, "$T$" means transposition.

Let

$$\mathcal{P} = \mathcal{L}^T \mathcal{L}/L,$$

$$q(\xi,t) = p(X,t),$$
$$\boldsymbol{h}(\xi,t) = \widehat{\mathcal{L}}\boldsymbol{H}(X,t), \quad \boldsymbol{e}(\xi,t) = \widehat{\mathcal{L}}\boldsymbol{E}(X,t),$$

i.e., $\boldsymbol{H}(X,t) = \frac{\mathcal{L}}{L}\boldsymbol{h}$, $\boldsymbol{E}(X,t) = \frac{\mathcal{L}}{L}\boldsymbol{e}$.

The mapping X converts (1.1)-(1.5) in

$$\begin{cases} \boldsymbol{u}_t - \nu\nabla_u^2\boldsymbol{u} + \nabla_u q - \nabla_u \cdot T_M(\mathcal{L}\boldsymbol{h}) = \boldsymbol{f}(X,t), \\ \nabla_u \cdot \boldsymbol{u} = 0, \quad \xi \in \Omega_1, \ \ t > 0, \end{cases} \tag{2.1}$$

$$\begin{cases} \mu(\boldsymbol{h}_t - \widehat{\mathcal{L}}_t^T \dfrac{\mathcal{L}}{L}\boldsymbol{h} - \widehat{\mathcal{L}}^T(\boldsymbol{u}\cdot\nabla_u)\dfrac{\mathcal{L}}{L}\boldsymbol{h}) = -rot\mathcal{P}(\xi,t)\boldsymbol{e}, \\ \xi \in \Omega_i, \quad i = 1,2,3, \\ \mathcal{P}rot\mathcal{P}\boldsymbol{h} = \alpha(\mathcal{P}\boldsymbol{e} + \mu(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h})), \ \nabla\cdot\boldsymbol{h} = 0, \ \xi \in \Omega_1, \\ rot\mathcal{P}\boldsymbol{h} = 0, \quad \nabla\cdot\boldsymbol{h} = 0, \quad \nabla\cdot\boldsymbol{e} = 0, \quad \xi \in \Omega_2, \\ rot\boldsymbol{h} = \alpha\boldsymbol{e} + \boldsymbol{j}(\xi,t), \quad \nabla\cdot\boldsymbol{h} = 0, \quad \xi \in \Omega_3, \end{cases} \tag{2.2}$$

$$\begin{cases} \boldsymbol{h}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{e}_\tau = 0, \quad \xi \in S, \\ [\mu\boldsymbol{h}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{h}_\tau] = 0, \quad [\boldsymbol{e}_\tau] = 0 \quad \xi \in S_3, \end{cases} \tag{2.3}$$

$$\begin{cases} [\mu\boldsymbol{h}\cdot\boldsymbol{n}_0] = 0, \quad [\boldsymbol{h}_\tau] = (\dfrac{A^T A\boldsymbol{n}_0}{|A\boldsymbol{n}_0|^2} - \boldsymbol{n}_0))[\boldsymbol{h}\cdot\boldsymbol{n}_0], \\ [\boldsymbol{n}_0 \times \mathcal{P}\boldsymbol{e}] = (\boldsymbol{u}\cdot A\boldsymbol{n}_0)[\mu]\boldsymbol{h}_\tau, \\ (T_u(\boldsymbol{u},q) + [T_M(\mathcal{L}\boldsymbol{h})])A\boldsymbol{n}_0 = 0, \quad \xi \in \Gamma_0, \end{cases} \tag{2.4}$$

$$\begin{cases} \boldsymbol{u}(\xi,0) = \boldsymbol{v}_0(\xi), \quad \xi \in \Omega_1, \\ \boldsymbol{h}(\xi,0) = \boldsymbol{H}_0(\xi), \quad \xi \in \Omega_i, \quad i = 1,2,3, \\ \displaystyle\int_{S_k} \boldsymbol{e}(\xi,t)\cdot\boldsymbol{n}(\xi)dS = 0, \end{cases} \tag{2.5}$$

where $\nabla_u = \mathcal{L}^{-T}\nabla$ is the transformed gradient w.r.to x and ∇ is the gradient w.r.to ξ, $\boldsymbol{n}(X) = A\boldsymbol{n}_0(\xi)/|A\boldsymbol{n}_0(\xi)|$, $\boldsymbol{n}_0$: the exterior normal to Γ_0.
$T_u = -q + \nu S_u(\boldsymbol{u})$: the transformed stress tensor;
$S_u(\boldsymbol{w}) = \nabla_u\boldsymbol{w} + (\nabla_u\boldsymbol{w})^T$: the transformed rate-of-strain tensor.
We find the solution of (2.1)-(2.5) in anisotropic Sobolev-Slobodetskii spaces $W_2^{r,r/2}$, $r > 0$.

Theorem 1. *Let* $\Gamma_0 \in W_2^{l+3/2}$, $\boldsymbol{u}_0 \in W_2^{l+1}(\Omega_1)$, $\boldsymbol{H}_0 \in W_2^l(\Omega_1) \cap W_2^l(\Omega_2)$, $\cap W_2^l(\Omega_3)$ *with* $3/2 < l < 2$, *and let the compatibility conditions (1.6) be satisfied. Assume also that*

$$\begin{aligned} &\boldsymbol{f}, \nabla\boldsymbol{f} \in W_2^{l,l/2}(\mathbb{R}^3 \times (0,T_0)), \\ &\boldsymbol{j} \in W_2^{(l-1)/2}(0,T_0; W_2^1(\Omega_3)) \cap L_2(0,T_0; W_2^l(\Omega_3)), \\ &\boldsymbol{j}(x,t)\cdot\boldsymbol{n}(x)|_{x\in S_3} = 0. \end{aligned} \tag{2.6}$$

Then the problem (2.1)-(2.5) has a unique solution defined in a certain (small) time interval $(0, T)$, $T \le T_0$ with the following regularity properties:

$$\begin{aligned}
&\boldsymbol{u} \in W_2^{2+l,1+l/2}(Q_T^1), \quad \nabla q \in W_2^{l,l/2}(Q_T^1),\\
&q \in W_2^{l+1/2,l/2+1/4}(G_T),\\
&\boldsymbol{h} \in W_2^{l+1,(l+1)/2}(Q_T^i),\\
&\boldsymbol{e} \in L_2(0,T; W_2^l(\Omega_i)) \cap W_2^{(l-1)/2}(0,T; W_2^1(\Omega_{0i})),
\end{aligned}$$

where $i = 1, 2, 3,$*,* $Q_T^i = \Omega \times (0,T)$*,* $G_T = \Gamma_0 \times (0,T)$*. The solution satisfies the inequality*

$$\begin{aligned}
&\|\boldsymbol{u}\|_{W_2^{l+2,l/2+1}(Q_T^1)} + \|\nabla q\|_{W_2^{l,l/2}(Q_T^1)}\\
&+ \|q\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\\
&+ \sum_{i=1}^{3} \big(\|\boldsymbol{h}\|_{W_2^{l+1,l/2+1/2}(Q_T^i)} + \|\boldsymbol{e}\|_{L_2(0,T;W_2^l(\Omega_i))}\\
&+ \|\boldsymbol{e}\|_{W_2^{(l-1)/2}(0,T;W_2^1(\Gamma_0))}\big)\\
&\le c\Big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T^1)} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_{01})}\\
&+ \sum_{i=1}^{3} \|\boldsymbol{H}_0\|_{W_2^l(\Omega_i)} + \|\boldsymbol{j}\|_{L_2(0,T;W_2^l(\Omega_3))}\\
&+ \|\boldsymbol{j}\|_{W_2^{(l-1)/2}(0,T;W_2^1(\Omega_3))}\Big).
\end{aligned} \tag{2.7}$$

Exclusion of $\boldsymbol{e}$. Let the test function $\boldsymbol{\psi}(\xi)$ satisfy the conditions

$$\begin{aligned}
&\nabla \cdot \boldsymbol{\psi}(\xi) = 0, \quad \xi \in \Omega_i, \quad i = 1, 2, 3,\\
&rot\boldsymbol{\psi}(\xi) = 0, \quad \xi \in \Omega_2,\\
&[\mu\boldsymbol{\psi} \cdot \boldsymbol{n}_0] = 0, \quad [\boldsymbol{\psi}_\tau] = 0, \quad \xi \in \Gamma_0,\\
&[\mu\boldsymbol{\psi} \cdot \boldsymbol{n}] = 0, \quad [\boldsymbol{\psi}_\tau] = 0, \quad \xi \in S_3,\\
&\boldsymbol{\psi} \cdot \boldsymbol{n} = 0, \quad \xi \in S.
\end{aligned} \tag{2.8}$$

Using the relation

$$\int_\Omega rot\mathcal{P}\boldsymbol{e} \cdot \boldsymbol{\psi} d\xi = \int_\Omega \mathcal{P}\boldsymbol{e} \cdot rot\boldsymbol{\psi} d\xi + \int_{\Gamma_0} [\boldsymbol{n}_0 \times \mathcal{P}\boldsymbol{e}] \cdot \boldsymbol{\psi} dS$$

we write (2.1)-(2.5) as a nonlinear problem for $\boldsymbol{u}, q, \boldsymbol{h}$:

$$
\begin{cases}
\boldsymbol{u}_t-\nu\nabla_u^2\boldsymbol{u}+\nabla_u q-\nabla_u\cdot T_M(\mathcal{L}\boldsymbol{h})=\boldsymbol{f}(X,t),\\
\nabla_u\cdot\boldsymbol{u}=0,\quad \xi\in\Omega_1,\ \ t>0,\\
\displaystyle\int_0^T\int_\Omega \mu(\boldsymbol{h}_t-\boldsymbol{\Phi}(\boldsymbol{h},\boldsymbol{u}))\cdot\boldsymbol{\psi}(\xi,t)d\xi dt\\
\displaystyle+\int_0^T\int_{\Omega_1\cup\Omega_3}\alpha^{-1}\mathcal{P}rot\mathcal{P}\boldsymbol{h}\cdot rot\boldsymbol{\psi}(\xi,t)d\xi dt\\
\displaystyle-\mu_1\int_0^T\int_{\Omega_1}(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h})\cdot rot\boldsymbol{\psi}(\xi,t)d\xi dt\\
\displaystyle+\int_0^T\int_{\Gamma_0}\boldsymbol{\Psi}(\boldsymbol{h},\boldsymbol{u})\cdot\boldsymbol{\psi}dSdt\\
\displaystyle=\int_0^T\int_{\Omega_3}\alpha^{-1}\boldsymbol{j}(\xi,t)\cdot\boldsymbol{\psi}(\xi,t)d\xi dt,\\
rot\mathcal{P}\boldsymbol{h}=0,\quad \xi\in\Omega_2,\\
\nabla\cdot\boldsymbol{h}=0,\quad \xi\in\Omega_1\cup\Omega_2\cup\Omega_3,\\
\boldsymbol{h}\cdot\boldsymbol{n}=0,\quad \xi\in S,\\
[\mu\boldsymbol{h}\cdot\boldsymbol{n}]=0,\quad [\boldsymbol{h}_\tau]=0,\quad \xi\in S_3,\\
[\mu\boldsymbol{h}\cdot\boldsymbol{n}_0]=0,\quad [\boldsymbol{h}_\tau]=(\dfrac{A^TA\boldsymbol{n}_0}{|A\boldsymbol{n}_0|^2}-\boldsymbol{n}_0))[\boldsymbol{h}\cdot\boldsymbol{n}_0)]=0,\\
(T_u(\boldsymbol{u},q)+[T_M(\mathcal{L}\boldsymbol{h})])A\boldsymbol{n}_0=0,\quad \xi\in\Gamma_0,\\
\boldsymbol{u}(\xi,0)=\boldsymbol{v}_0(\xi),\quad \xi\in\Omega_1,\\
\boldsymbol{h}(\xi,0)=\boldsymbol{H}_0(\xi),\quad \xi\in\Omega_i,\quad i=1,2,3,
\end{cases}
\tag{2.9}
$$

where

$$
\boldsymbol{\Phi}=\widehat{\mathcal{L}}_t^T\frac{\mathcal{L}}{L}\boldsymbol{h}+\widehat{\mathcal{L}}^T(\boldsymbol{u}\cdot\nabla_u)\frac{\mathcal{L}}{L}\boldsymbol{h},\quad \boldsymbol{\Psi}=(\boldsymbol{u}\cdot A\boldsymbol{n}_0)[\mu]\boldsymbol{h}_\tau.
$$

Theorem 2. *Under the assumptions of Theorem 1, the problem (2.9) has a unique solution* $(\boldsymbol{u},q,\boldsymbol{h})$ *satisfying the inequality*

$$
\begin{aligned}
&\|\boldsymbol{u}\|_{W_2^{l+2,l/2+1}(Q_T^1)}+\|\nabla q\|_{W_2^{l,l/2}(Q_T^1)}\\
&+\|q\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\\
&+\sum_{i=1}^3\big(\|\boldsymbol{h}\|_{W_2^{l+1,l/2+1/2}(Q_T^i)}\\
&\le c\Big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T^1)}+\|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_{01})}\\
&+\sum_{i=1}^3\|\boldsymbol{H}_0\|_{W_2^l(\Omega_i)}+\|\boldsymbol{j}\|_{L_2(0,T;W_2^l(\Omega_3))}\\
&+\|\boldsymbol{j}\|_{W_2^{(l-1)/2}(0,T;W_2^1(\Omega_3))}\Big).
\end{aligned}
\tag{2.10}
$$

3 Linear problems

The proof of Theorem 2 is based on the analysis of some linear problems, namely,

$$\begin{cases} \boldsymbol{v}_t - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(\xi,t), \ \ \nabla\cdot\boldsymbol{v} = f, \quad \xi\in\Omega_1, \ \ t>0, \\ T(\boldsymbol{v},p)\boldsymbol{n} = \boldsymbol{d}(\xi,t), \ \ \xi\in\Gamma_0, \\ \boldsymbol{v}(\xi,0) = \boldsymbol{v}_0(\xi), \quad \xi\in\Omega_1, \end{cases} \tag{3.1}$$

$$\begin{cases} rot\boldsymbol{h}(\xi) = \boldsymbol{k}(x), \quad \nabla\cdot\boldsymbol{h}(\xi) = 0, \quad \xi\in\Omega_1\cup\Omega_2\cup\Omega_3, \\ [\mu\boldsymbol{h}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{h}_\tau] = 0, \quad \xi\in\Gamma_0\cup S_3, \quad \boldsymbol{h}\cdot\boldsymbol{n}|_{\xi\in S} = 0, \end{cases} \tag{3.2}$$

$$\begin{cases} \int_0^T\int_\Omega \mu\boldsymbol{h}_t\cdot\boldsymbol{\psi}(\xi,t)d\xi dt \\ +\int_{\Omega_1\cup\Omega_3}\alpha^{-1}rot\boldsymbol{h}\cdot rot\boldsymbol{\psi}(\xi,t)d\xi dt \\ =\int_0^T\int_{\Omega_1\cup\Omega_3}\boldsymbol{g}_1(\xi,t)\cdot rot\boldsymbol{\psi}(y,t)d\xi dt \\ +\int_0^T\int_\Omega \mu\boldsymbol{g}_2(y,t)\cdot\boldsymbol{\psi}(\xi,t)d\xi dt, \\ \nabla\cdot\boldsymbol{h}(\xi,t) = 0, \quad \xi\in\Omega_1\cup\Omega_2\cup\Omega_3, \\ rot\boldsymbol{h}(\xi,t) = 0, \quad \xi\in\Omega_2, \\ [\mu\boldsymbol{h}\cdot\boldsymbol{n}_0] = 0, \quad [\boldsymbol{h}_\tau] = 0, \quad \xi\in\Gamma_0, \\ [\mu\boldsymbol{h}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{h}_\tau] = 0, \quad \xi\in S_3, \\ \boldsymbol{h}\cdot\boldsymbol{n} = 0, \quad \xi\in S, \\ \boldsymbol{h}(\xi,0) = \boldsymbol{h}_0(\xi), \quad \xi\in\Omega_1\cup\Omega_2\cup\Omega_3, \end{cases} \tag{3.3}$$

where $\boldsymbol{\psi}(\xi,t)$ is an arbitrary test vector field satisfying the conditions (2.8).

Theorem 3. *Assume that* $\Gamma_0\in W_2^{l+3/2}$, $3/2<l<2$,

$$\begin{aligned} &\boldsymbol{f}\in W_2^{l,l/2}(Q_T^1), \quad \boldsymbol{v}_0\in W_2^{l+1}(\Omega_1), \\ &f = \nabla\cdot\boldsymbol{F}, \quad f\in L_2(0,T;W_2^{l+1}(\Omega_1)), \\ &\boldsymbol{F}_t\in W_2^{l/2}(0,T;L_2(\Omega_1)), \\ &\boldsymbol{d}\in W_2^{l+1/2,l/2+1/4}(G_T), \end{aligned}$$

and that the compatibility conditions

$$\nabla\cdot\boldsymbol{v}_0(\xi) = f(\xi,0), \quad (S(\boldsymbol{v}_0)\boldsymbol{n})_\tau = \boldsymbol{d}_\tau, \quad \xi\in\Gamma_0$$

are satisfied. Then the problem (3.1) has a unique solution $\boldsymbol{v}\in W_2^{2+l,1+l/2}(Q_T^1)$, $\nabla p\in W_1^{l,l/2}(Q_T^1)$ *with* $p|_{\xi\in\Gamma_0}\in W_2^{l+1/2,l/2+1/4}(G_T)$ *and*

$$\begin{aligned} &\|\boldsymbol{v}\|_{W_2^{l+2,l/2+1}(Q_T^1)} + \|\nabla p\|_{W_2^{l,l/2}(Q_T^1)} + \|p\|_{W_2^{l+l/2,1/2+l/4}(\Gamma_0)} \\ &\le c(\|f\|_{L_2(0,T;W_2^{l+1}(\Omega_1))} + \|\boldsymbol{F}_t\|_{W_2^{l/2}(0,T;L_2(\Omega_1))} \\ &+ \|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_0)}). \end{aligned} \tag{3.4}$$

Let $\mathcal{H}^l(\Omega)$ be the space of vector fields $\boldsymbol{\psi} \in W_2^l(\Omega)$, $i = 1, 2, 3$, satisfying (2.8);

$\mathfrak{U}(\Omega)$: finite dimensional set of vector fields satisfying (2.8) and, in addition, $rot\boldsymbol{\psi}(\xi) = 0$, $\xi \in \Omega_1 \cup \Omega_2 \cap \Omega_3$;

$\mathcal{H}^l_\perp(\Omega)$: the subset of $\mathcal{H}^l(\Omega)$ whose elements satisfy the orthogonality condition

$$\int_\Omega \mu\boldsymbol{\psi}(\xi) \cdot \boldsymbol{u}(\xi) d\xi = 0, \quad \forall \boldsymbol{u} \in \mathfrak{U}(\Omega).$$

Theorem 4. *Let* $\Gamma_0, S, S_3 \in W_2^{l+3/2}$, $l \in (3/2, 2)$. *For arbitrary* $\boldsymbol{k} \in W_2^r(\Omega_i)$, $r \in [0, l]$ $i = 1, 2, 3$, *such that*

$$\begin{aligned}
&\nabla \cdot \boldsymbol{k} = 0, \quad \xi \in \Omega_1 \cup \Omega_2 \cup \Omega_3, \\
&[\boldsymbol{k} \cdot \boldsymbol{n}] = 0, \quad \xi \in \Gamma_0 \cup S_3, \quad \boldsymbol{k} \cdot \boldsymbol{n}|_S = 0
\end{aligned}$$

the problem (3.2) has a unique solution belonging to $W_2^{r+1}(\Omega_i)$, $i = 1, 2, 3$ *and orthogonal to* $\mathfrak{U}(\Omega)$. *The solution satisfies the inequality*

$$\sum_{i=1}^{3} \|\boldsymbol{h}\|_{W_2^{1+r}(\Omega_i)} \le c \sum_{i=1}^{3} \|\boldsymbol{k}\|_{W_2^r(\Omega_i)}. \tag{3.5}$$

Moreover, if

$$\boldsymbol{k} = rot\boldsymbol{K}(\xi), \quad [\boldsymbol{K}_\tau]|_{\Gamma_0 \cup S_3} = 0, \quad \boldsymbol{K}_\tau|_S = 0, \tag{3.6}$$

then

$$\|\boldsymbol{h}\|_{L_2(\Omega)} \le c\|\boldsymbol{K}\|_{L_2(\Omega)}. \tag{3.7}$$

Corollary 1. *If* $\boldsymbol{h} \in \mathcal{H}^{r+1}_\perp(\Omega)$, *then*

$$\begin{aligned}
&c_1 \sum_{i=1}^{3} \|\boldsymbol{h}\|_{W_2^{r+1}(\Omega_i)} \le \|rot\boldsymbol{h}\|_{W_2^r(\Omega_1 \cup \Omega_3)} \\
&\le c_2 \sum_{i=1}^{3} \|\boldsymbol{h}\|_{W_2^{r+1}(\Omega_i)}.
\end{aligned}$$

Corollary 2. *If* $\boldsymbol{k} \in W_2^{r,r/2}(Q_T^i)$, $i = 1, 2, 3$, *then* $\boldsymbol{h} \in L_2(0, T : W_2^{r+1}(\Omega_i)) \cap W_2^{r/2}(0, T; W_2^1(\Omega_i))$ *and*

$$\begin{aligned}
&\sum_{i=1}^{3} (\|\boldsymbol{h}\|_{L_2(0,T:W_2^{r+1}(\Omega_i))} + \|\boldsymbol{h}\|_{W_2^{r/2}(0,T;W_2^1(\Omega_i))}) \\
&\le c\|\boldsymbol{k}\|_{W_2^{r,r/2}(\Omega_i)}.
\end{aligned} \tag{3.8}$$

Moreover, if (3.6) holds, then

$$\|\boldsymbol{h}\|_{W_2^{(r+1)/2}(0,T;L_2(\Omega))} \le c\|\boldsymbol{K}\|_{W_2^{(r+1)/2}(0,T;L_2(\Omega))}. \tag{3.9}$$

Theorem 5. *Let* $l \in (3/2, 2)$. *For arbitrary* $\boldsymbol{h}_0$, $\boldsymbol{g}_1$, $\boldsymbol{g}_2$ *satisfying the conditions*

$$\begin{aligned}
&\boldsymbol{h}_0 \in W_2^l(\Omega_i), \quad i = 1, 2, 3, \\
&\nabla \cdot \boldsymbol{h}_0(y) = 0, \quad y \in \Omega_1 \cup \Omega_2 \cup \Omega_3, \\
&rot\boldsymbol{h}_0(y) = 0, \quad y \in \Omega_2, \\
&[\mu\boldsymbol{h}_0 \cdot \boldsymbol{n}] = 0, \quad [\boldsymbol{h}_{0\tau}] = 0, \quad y \in \Gamma_0 \cup S_3, \\
&\boldsymbol{h}_0 \cdot \boldsymbol{n} = 0, \quad y \in S,
\end{aligned} \tag{3.10}$$

$$
\begin{aligned}
&\boldsymbol{g}_1 \in L_2(0,T;W_2^l(\Omega_1\cup\Omega_3))\\
&\cap W_2^{(l-1)/2}(0,T;W_2^1(\Omega_1\cup\Omega_3)),\\
&\nabla\cdot\boldsymbol{g}_1=0,\quad \boldsymbol{g}_1\cdot\boldsymbol{n}=0,\quad y\in\Gamma_0\cup S_3,\\
&\boldsymbol{g}_2\in W_2^{l-1,(l-1)/2}(Q_T^i),\quad i=1,2,3,\\
&[\mu\boldsymbol{g}_2\cdot\boldsymbol{n}]=0,\quad [\boldsymbol{g}_{2\tau}]=0,\quad y\in\Gamma_0\cup S_3,\\
&\boldsymbol{g}_2\cdot\boldsymbol{n}=0,\quad y\in S
\end{aligned}
\tag{3.11}
$$

the problem (3.3) has a unique solution
$\boldsymbol{h}\in W_2^{1+l,(1+l)/2}(Q_T^i)$, *satisfying the inequality*

$$
\begin{aligned}
&\sum_{i=1}^{3}\|\boldsymbol{h}\|_{W_2^{1+l,(1+l)/2}(Q_T^i)}\\
&\le c\Big(\sum_{i=1}^{3}(\|\boldsymbol{h}_0\|_{W_2^l(\Omega_i)}+\|\boldsymbol{g}_2\|_{W_2^{l-1,(l-1)/2}(Q_T^i)})\\
&+\sum_{i=1,3}(\|\boldsymbol{g}_1\|_{L_2(0,T;W_2^l(\Omega_i))}\\
&+\|\boldsymbol{g}_1\|_{W_2^{(l-1)/2}(0,T;W_2^1(\Omega_i))})\Big),
\end{aligned}
\tag{3.12}
$$

Moreover, there exist $\boldsymbol{e}\in L_2(0,T;W_2^l(\Omega_i))\cap W_2^{(l-1)/2}(0,T;W_2^1(\Omega_i))$, $i=1,2,3$, *such that*

$$
\begin{cases}
\mu(\boldsymbol{h}_t-\boldsymbol{g}_2)=-rot\boldsymbol{e}(\xi,t),\quad \xi\in\Omega_1\cup\Omega_2\cup\Omega_3,\\
rot\boldsymbol{h}=\alpha(\boldsymbol{e}+\boldsymbol{g}_1).\quad \xi\in\Omega_1\cup\Omega_3,\\
\nabla\cdot\boldsymbol{e}(\xi,t)=0,\quad \xi\in\Omega_2,\\
[\boldsymbol{e}_\tau]=0,\quad \xi\in\Gamma_0\cup S_3,\quad \boldsymbol{e}_\tau=0,\quad \xi\in S,
\end{cases}
\tag{3.13}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{3}(\|\boldsymbol{e}\|_{L_2(0,T;W_2^l(\Omega_i))}+\|\boldsymbol{e}\|_{W_2^{(l-1)/2}(0,T;W_2^1(\Omega_i))})\\
&\le c\Big(\sum_{i=1}^{3}(\|\boldsymbol{h}_0\|_{W_2^l(\Omega_i)}+\|\boldsymbol{g}_2\|_{W_2^{l-1,(l-1)/2}(Q_T^i)})\\
&+\sum_{i=1,2}(\|\boldsymbol{g}_1\|_{L_2(0,T;W_2^l(\Omega_i))}+\|\boldsymbol{g}_1\|_{W_2^{(l-1)/2}(0,T;W_2^1(\Omega_i))})\Big).
\end{aligned}
\tag{3.14}
$$

We describe briefly the (formal) construction of $\boldsymbol{e}$. We decompose $\boldsymbol{h},\boldsymbol{h}_0,\boldsymbol{\psi},\boldsymbol{g}_2$ in the sums

$$
\boldsymbol{h}=\boldsymbol{h}'+\boldsymbol{h}'',\quad \boldsymbol{h}_0=\boldsymbol{h}_0'+\boldsymbol{h}_0'',
$$

$$
\boldsymbol{\psi}=\boldsymbol{\psi}'+\boldsymbol{\psi}'',\quad \boldsymbol{g}_2=\boldsymbol{g}_2'+\boldsymbol{g}_2'',
$$

where $\boldsymbol{h}'',\boldsymbol{h}_0'',\boldsymbol{\psi}'',\boldsymbol{g}_2''\in\mathfrak{U}(\Omega)$ and $\boldsymbol{h}',\boldsymbol{h}_0',\boldsymbol{\psi}',\boldsymbol{g}_2'$ are orthogonal to $\mathfrak{U}(\Omega)$. It is easily verified that in the case $\boldsymbol{h}_0=\boldsymbol{h}_0''$, $\boldsymbol{g}_2=\boldsymbol{g}_2''$ the problem (3.3) reduces to

$$
\int_0^T\int_\Omega \mu(\boldsymbol{h}_t''-\boldsymbol{g}_2'')\cdot\boldsymbol{\psi}''\,d\xi dt=0,
$$

i.e.,

$$\boldsymbol{h}_t^{''} - \boldsymbol{g}_2^{''} = 0, \quad \boldsymbol{h}^{''}|_{t=0} = \boldsymbol{h}_0^{''}, \tag{3.15}$$

and in the case $\boldsymbol{h}_0 = \boldsymbol{h}_0^{'}, \quad \boldsymbol{g}_2 = \boldsymbol{g}_2^{'}$ we have $\boldsymbol{h} = \boldsymbol{h}^{'}$ [1]. Let $\mathcal{E}$ be the solution of the problem

$$\begin{aligned} &rot\boldsymbol{\mathcal{E}} = -\mu(\boldsymbol{h}_t' - \boldsymbol{g}_2'), \quad \nabla\cdot\boldsymbol{\mathcal{E}} = 0, \quad \xi\in\Omega, \\ &\boldsymbol{\mathcal{E}}_\tau = 0, \quad \xi\in S. \end{aligned} \tag{3.16}$$

It is easily seen that

$$\int_0^T \int_{\Omega_1\cup\Omega_3} (-\boldsymbol{\mathcal{E}} + \alpha^{-1} rot\boldsymbol{h}' - \boldsymbol{g}_1)\cdot rot\boldsymbol{\psi}'(y,t)d\xi dt = 0,$$

which implies

$$-\boldsymbol{\mathcal{E}} + \alpha^{-1} rot\boldsymbol{h} - \boldsymbol{g}_1 = \nabla\chi_1(\xi,t), \quad \xi\in\Omega_1\cup\Omega_3, \tag{3.17}$$

in view of Theorem 4. We set

$$\begin{aligned} &\boldsymbol{e} = \boldsymbol{\mathcal{E}} + \nabla\chi_1(\xi,t), \quad \xi\in\Omega_1\cup\Omega_3, \\ &\boldsymbol{e} = \boldsymbol{\mathcal{E}} + \nabla\chi_2(\xi,t) + \sum_{k=1}^{b} C_j(t)\boldsymbol{v}_k(\xi), \quad \xi\in\Omega_2, \end{aligned} \tag{3.18}$$

where χ_2 is a solution of the problem

$$\begin{aligned} &\nabla^2\chi_2(\xi,t) = 0, \quad \xi\in\Omega_2, \\ &\chi_2(\xi,t) = \chi_1(\xi,t), \quad \xi\in\Gamma_0\cup S_3, \quad \chi_2(\xi,t) = 0, \quad \xi\in S \end{aligned} \tag{3.19}$$

and $\boldsymbol{v}_k$ are the "Dirichlet vector fields" in Ω_2, i.e., $\boldsymbol{v}_k = \nabla\phi(\xi,t)$,

$$\begin{aligned} &\nabla^2\phi_k(\xi,t) = 0, \quad \xi\in\Omega_2, \\ &\phi_k(\xi,t) = \delta_{jk}, \quad \xi\in S_j, \quad \phi_k(\xi,t) = 0, \quad \xi\in S, \end{aligned} \tag{3.20}$$

S_j are all the connected components of $\partial\Omega_2$, except S, $j,k = 1,...,b$, b is the second Betti number of the domain Ω_2.

The coefficients $C_j(t)$ may be found from the additional normalization conditions

$$\int_{S_k} \boldsymbol{e}\cdot\boldsymbol{n}dS = 0,$$

that gives

$$0 = \int_{S_k} (\boldsymbol{\mathcal{E}} + \nabla\chi_2(\xi,t) + \sum_{k=1}^{b} C_j(t)\boldsymbol{v}_k(\xi))\cdot\boldsymbol{n}dS. \tag{3.21}$$

The estimate (3.14) follows from (3.21)and from the estimates of solutions of (3.15), (3.16), (3.19), (3.20).

4 Nonlinear problem

We go back to Sec. 2 and write our main nonlinear problem in the form

$$\begin{cases} \boldsymbol{u}_t - \nu\nabla^2\boldsymbol{u} + \nabla q = \boldsymbol{f}(\xi, t) + \boldsymbol{l}_1(u, q), \\ \nabla \cdot \boldsymbol{u} = l_2(\boldsymbol{u}), \\ \xi \in \Omega_1, \ \ t > 0, \\ T(\boldsymbol{u}, q)\boldsymbol{n} = \boldsymbol{l}_3(\boldsymbol{u}), \quad \xi \in \Gamma_0, \\ \int_0^T \int_\Omega \mu\boldsymbol{h}_t \cdot \boldsymbol{\psi}(y, t) dy dt \\ + \int_{\Omega_1 \cup \Omega_3} \alpha^{-1} rot\boldsymbol{h} \cdot rot\boldsymbol{\psi}(y, t) dy dt \\ = \int_0^T \int_{\Omega_1 \cup \Omega_3} \boldsymbol{l}_4(\boldsymbol{u}, \boldsymbol{h}) \cdot rot\boldsymbol{\psi}(y, t) dy dt \\ + \int_0^T \int_\Omega \mu\boldsymbol{l}_5(\boldsymbol{u}, \boldsymbol{h}) \cdot \boldsymbol{\psi}(y, t) dy dt, \\ rot\boldsymbol{h} = \boldsymbol{l}_6(\boldsymbol{u}, \boldsymbol{h}), \quad \xi \in \Omega_2, \\ [\mu\boldsymbol{h} \cdot \boldsymbol{n}] = 0, \ \ [\boldsymbol{h}_\tau] = \boldsymbol{l}_7(\boldsymbol{u}, \boldsymbol{h}), \ \ \xi \in \Gamma_0, \\ [\mu\boldsymbol{h} \cdot \boldsymbol{n}] = 0, \quad [\boldsymbol{h}_\tau] = 0, \ \ \xi \in S_3, \ \ \boldsymbol{h} \cdot \boldsymbol{n} = 0, \quad \xi \in S, \\ \boldsymbol{u}(\xi, 0) = \boldsymbol{v}_0(\xi), \quad \xi \in \Omega_1, \\ \boldsymbol{h}(\xi, 0) = \boldsymbol{H}_0(\xi), \quad \xi \in \Omega_i, \quad i = 1, 2, 3, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} &\boldsymbol{l}_1 = \nu(\nabla_u^2 - \nabla^2)\boldsymbol{u} + (\nabla - \nabla_u)q + \nabla_u \cdot T_M(\mathcal{L}\boldsymbol{h}) - \boldsymbol{f}(X, t) + \boldsymbol{f}(\xi, t), \\ &\boldsymbol{l}_2 = (\nabla - \nabla_u) \cdot \boldsymbol{h} = \nabla \cdot (I - A^T)\boldsymbol{h}, \\ &\boldsymbol{l}_3 = \Pi_0(\Pi_0 S(\boldsymbol{u})\boldsymbol{n}_0 - \Pi S_u(\boldsymbol{u})\boldsymbol{n}) \\ &\ + \nu\boldsymbol{n}_0((\boldsymbol{n}_0 \cdot S(\boldsymbol{u})\boldsymbol{n}_0) - (\boldsymbol{n} \cdot S_u(\boldsymbol{u})\boldsymbol{n})) + \boldsymbol{n}_0[\boldsymbol{n} \cdot T_M(\mathcal{L}\boldsymbol{h})\boldsymbol{n}], \\ &\boldsymbol{l}_4 = \alpha^{-1}(rot\boldsymbol{h} - \mathcal{P}rot\mathcal{P}\boldsymbol{h}) + \mu(\mathcal{L}^{-1}\boldsymbol{u} \times \boldsymbol{h}) - (\boldsymbol{n}_0^* \times \boldsymbol{\Psi}^*), \\ &\boldsymbol{l}_5 = \boldsymbol{\Phi}(\boldsymbol{u}, \boldsymbol{h}) + \mu^{-1} rot(\boldsymbol{n}_0^* \times \boldsymbol{\Psi}^*), \\ &\boldsymbol{l}_6 = rot(I - \mathcal{P})\boldsymbol{h}, \\ &\boldsymbol{l}_7 = (\frac{A^T A\boldsymbol{n}_0}{|\boldsymbol{A}\boldsymbol{n}_0|^2} - \boldsymbol{n}_0)[\boldsymbol{h} \cdot \boldsymbol{n}_0], \end{aligned} \tag{4.2}$$

where $\boldsymbol{n}_0^*$ and $\boldsymbol{\Psi}^*$ are extensions of $\boldsymbol{n}_0$, $\boldsymbol{\Psi}$ from Γ_0 in Ω_1,

$$\Pi\boldsymbol{f} = \boldsymbol{f} - \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{f}), \quad \Pi_0\boldsymbol{f} = \boldsymbol{f} - \boldsymbol{n}_0(\boldsymbol{n}_0 \cdot \boldsymbol{f}).$$

We solve the problem (4.1) by successive approximations, according to the scheme

$$
\begin{cases}
\boldsymbol{u}_{m+1,t} - \nu\nabla^2\boldsymbol{u}_{m+1} + \nabla q_{m+1} = \boldsymbol{f}(\xi,t) + \boldsymbol{l}_1(u_m, q_m), \\
\nabla\cdot\boldsymbol{u}_{m+1} = l_2(\boldsymbol{u}_m), \\
\xi\in\Omega_{10}, \quad t>0, \\
T(\boldsymbol{u}_{m+1}, q_{m+1})\boldsymbol{n} = \boldsymbol{l}_3(\boldsymbol{u}_m, \boldsymbol{h}_m), \quad \xi\in\Gamma_0, \\
\int_0^T\int_\Omega \mu\boldsymbol{h}_{m+1t}\cdot\boldsymbol{\psi}(y,t)dydt \\
+\int_{\Omega_1\cup\Omega_3}\alpha^{-1} rot\boldsymbol{h}_{m+1}\cdot rot\boldsymbol{\psi}(y,t)dydt \\
=\int_0^T\int_{\Omega_1\cup\Omega_3}\boldsymbol{l}_4(\boldsymbol{u}_m,\boldsymbol{h}_m)\cdot rot\boldsymbol{\psi}(y,t)dydt \\
+\int_0^T\int_\Omega \mu\boldsymbol{l}_5(\boldsymbol{u}_m,\boldsymbol{h}_m)\cdot\boldsymbol{\psi}(y,t)dydt, \\
rot\boldsymbol{h}_{m+1} = \boldsymbol{l}_6(\boldsymbol{u}_m,\boldsymbol{h}_m), \quad \xi\in\Omega_2, \\
[\mu\boldsymbol{h}_{m+1}\cdot\boldsymbol{n}]=0, \;\; [\boldsymbol{h}_{m+1,\tau}] = \boldsymbol{l}_7(\boldsymbol{u}_m,\boldsymbol{h}_m), \;\; \xi\in\Gamma_0, \\
[\mu\boldsymbol{h}_{m+1}\cdot\boldsymbol{n}]=0, \quad [\boldsymbol{h}_{m+1,\tau}]=0, \;\; \xi\in S_3, \;\; \boldsymbol{h}_{m+1}\cdot\boldsymbol{n}=0, \quad \xi\in S, \\
\boldsymbol{u}_{m+1}(\xi,0) = \boldsymbol{v}_0(\xi), \quad \xi\in\Omega_1, \\
\boldsymbol{h}_{m+1}(\xi,0) = \boldsymbol{H}_0(\xi), \quad \xi\in\Omega_i, \quad i=1,2,3,
\end{cases}
\tag{4.3}
$$

$\boldsymbol{u}_0=0$, $q_0=0$, $\rho_0=0$, $\boldsymbol{h}_0=0$. Using Theorems 3-5 and estimates of the nonlinear terms (they are omitted), we obtain

$$
Y_{m+1}(T) \le c_1T^\beta\sum_{j=1}^3 Y_m^j(t) + c_2N(T), \tag{4.4}
$$

where $\beta>0$,

$$
\begin{aligned}
&Y_m(T) = \|\boldsymbol{u}_m\|_{W_2^{l+2,l/2+1}(Q_T^1)} + \|\nabla q_m\|_{W_2^{l,l/2}(Q_T^1)} \\
&+\|q_m\|_{W_2^{l+1/2,l/2+1/4}(G_T)} \\
&+\sum_{i=1}^3\|\boldsymbol{h}_m\|_{W_2^{l+1,l/2+1/2}(Q_T^i)}, \\
&N(T) = \|\boldsymbol{f}\|_{W_2^{l,l/2}(W_T^i)} + \|\boldsymbol{j}\|_{L_2(0,T;W_2^l(\Omega_3))} \\
&+\|\boldsymbol{j}\|_{W_2^{(l-1)/2}(0,T;W_2^1(\Omega_3))} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_1)} \\
&+\sum_{i=1}^3\|\boldsymbol{h}_0\|_{W_2^l(\Omega_i)}.
\end{aligned}
$$

It follows from (4.4) that in the case of small T

$$
Y_{m+1}(T) \le 2c_2N(T). \tag{4.5}
$$

The proof of the convergence of the sequence $(\boldsymbol{u}_m, q_m, \rho_m, \boldsymbol{h}_m)$ is based on the estimate of the differences $(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m, q_{m+1} - q_m, \rho_{m+1} - \rho_m, \boldsymbol{h}_{m+1} - \rho_m)$, as in [2]. This yields the proof of Theorem 2. The reconstruction of $\boldsymbol{e}$ can be made in the same manner as in the linear case above, which completes the proof of Theorem 1.

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