Finite element analysis of a Cahn-Hilliard equation on an evolving surface

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- We consider the problem of surface dissolution.
- Namely, we look at dealloying of a binary alloy by the selective removal of one component via electrochemical dissolution in an electrolyte such that there is surface phase separation of the other component.
- A prototypical example is that of the etching of silver in an Ag–Au alloy whose surface is immersed in an electrolyte.

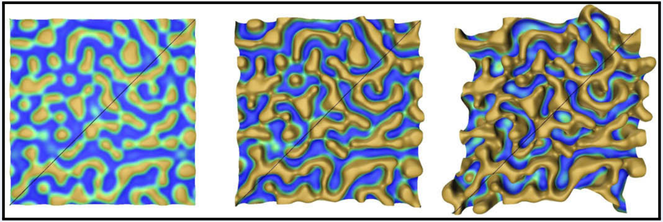

Figure: Taken from [1]

[1] Eilks and Elliott 2008

Find a periodic evolving surface $\Gamma(t)$, $t \in (0, T)$ and a periodic function $c\colon \bigcup_{t\in[0,T]} \Gamma(t) \times \{t\} \to \mathbb{R}$ such that the surface moves with velocity

$$V = v_0(c)(1 - \delta H)\nu,$$

and c solves the equation

$$\begin{aligned} \partial^\bullet c + c\nabla_\Gamma \cdot V - \nabla_\Gamma\big(b(c)\nabla w\big) &= c_0 V \cdot \nu \\ -\gamma\Delta_\Gamma c + \psi'(c) &= w. \end{aligned}$$

Simplifications

We will assume:

- velocity V known!
- constant mobility: $b(c) = 1$
- no bulk forcing
- closed surface

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Given $\{\Gamma(t)\}$, $t \in [0, T]$ find a function $u\colon \bigcup_{t\in(0,T)} \Gamma(t) \times \{t\} \to \mathbb{R}$ such that u satisfies

$$\partial^\bullet c + c\nabla_\Gamma \cdot V + \Delta_\Gamma w = 0$$
$$-\varepsilon\Delta_\Gamma u + \frac{1}{\varepsilon}\psi'(u) = w.$$

Outline

We define the material derivative of u by

$$\partial^{\bullet} u = \partial_t u + v \cdot \nabla u.$$

Lemma (Transport Lemma[2])

Assume u is a function such that all the following quantities exist.

$$\frac{d}{dt} \int_{\Gamma(t)} u = \int_{\Gamma(t)} \partial^{\bullet} u + u \nabla_{\Gamma} \cdot v.$$

[2] Dziuk and Elliott 2007

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The conservation law

$$\frac{d}{dt}\int_{\Gamma(t)} u = -\int_{\Gamma(t)} \nabla_\Gamma \cdot q$$

implies

$$\partial^\bullet u + u\nabla_\Gamma \cdot v + \nabla_\Gamma \cdot q = 0.$$

[2] Dziuk and Elliott 2007

We take $q = -\nabla_\Gamma w$ with w the chemical potential given by

$$w = -\varepsilon \Delta_\Gamma u + \frac{1}{\varepsilon}\psi'(u).$$

This leads to the system

$$\partial^\bullet u + u \nabla_\Gamma \cdot v - \Delta_\Gamma w = 0 \tag{1a}$$

$$-\varepsilon \Delta_\Gamma u + \frac{1}{\varepsilon}\psi'(u) - w = 0. \tag{1b}$$

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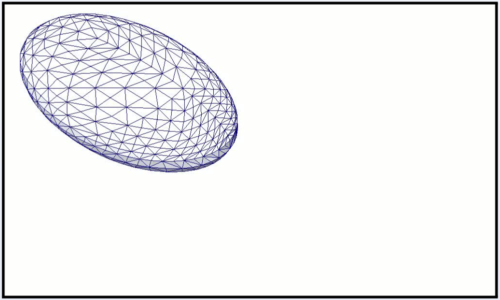
- mass is conserved: $\frac{d}{dt}\int_{\Gamma(t)} u = 0$
- The Ginzburg-Landau functional does not, in general, decrease.

In the case $v = 0$, we simply take a weak form and put h everywhere!

$$\int_{\Gamma_h} \partial_t U_h \phi_h + \int_{\Gamma_h} \nabla_{\Gamma_h} W_h \cdot \nabla_{\Gamma_h} \phi_h = 0$$

$$\varepsilon \int_{\Gamma_h} \nabla_{\Gamma_h} U_h \cdot \nabla_{\Gamma_h} \phi_h + \frac{1}{\varepsilon} \psi'(U_h) \phi_h - \int_{\Gamma_h} W_h \phi_h = 0$$

for all $\phi_h \in S_h$.



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$$\text{for all } \phi_h \in S_h.$$

Try the same:

$$\int_{\Gamma_h(t)} \partial_h^\bullet U_h \phi_h + \int_{\Gamma_h(t)} U_h \phi_h \nabla_{\Gamma_h} \cdot V_h + \int_{\Gamma_h(t)} \nabla_{\Gamma_h} W_h \cdot \nabla_{\Gamma_h(t)} \phi_h = 0$$

$$\varepsilon \int_{\Gamma_h(t)} \nabla_{\Gamma_h} U_h \cdot \nabla_{\Gamma_h} \phi_h + \frac{1}{\varepsilon}\psi'(U_h)\phi_h - \int_{\Gamma_h(t)} W_h \phi_h = 0$$

$$\text{for all } \phi_h \in S_h(t).$$

Discrete material derivative

This gives rise to a discrete material velocity V_h, only chosen so that the nodes of $\Gamma_h(t)$ lies on $\Gamma(t)$, and a discrete material derivative

$$\partial_h^\bullet U_h = \partial_t U_h + V_h \cdot \nabla U_h.$$

Then:

$$\frac{d}{dt}\int_{\Gamma_h(t)} U_h = \int_{\Gamma_h(t)} \partial_h^\bullet U_h + U_h \nabla_{\Gamma_h} \cdot V_h.$$

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This let's us define the variational form.

$$\frac{d}{dt} \int_{\Gamma_h(t)} U_h \phi_h + \int_{\Gamma_h(t)} \nabla_{\Gamma_h} W_h \cdot \nabla_{\Gamma_h(t)} \phi_h = \int_{\Gamma_h(t)} U_h \partial_h^\bullet \phi_h$$

$$\varepsilon \int_{\Gamma_h(t)} \nabla_{\Gamma_h} U_h \cdot \nabla_{\Gamma_h} \phi_h + \frac{1}{\varepsilon} \psi'(U_h) \phi_h - \int_{\Gamma_h(t)} W_h \phi_h = 0$$

$$\text{for all } \phi_h \in S_h(t). \tag{2}$$

Lemma (Transport of basis functions)

Let $\{\phi_j(\cdot, t)\}$ *be a basis for* $S_h(t)$ *then*

$$\partial_h^\bullet \phi_j = 0.$$

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The system (2) is equivalence to the matrix problem:

$$\frac{d}{dt}(\mathcal{M}(t)U(t)) + \mathcal{S}(t)W(t) = 0$$
$$\varepsilon \mathcal{S}(t)U(t) + \frac{1}{\varepsilon}\Psi(U(t)) - \mathcal{M}(t)W(t) = 0.$$

with

$$\mathcal{M}(t)_{ij} = \int_{\Gamma_h(t)} \phi_i \phi_j, \quad \mathcal{S}(t)_{ij} = \int_{\Gamma_h(t)} \nabla_{\Gamma_h}\phi_i \cdot \nabla_{\Gamma_h}\phi_j,$$
$$\Psi(\alpha)_j = \int_{\Gamma_h(t)} \psi'(\alpha(t))\phi_j.$$

Theorem (Stability)

There exists a unique solution pair $U_h, W_h \in S_h(t)$ that satisfy (2) which satisfy the bounds

$$\sup_t \left(\frac{\varepsilon}{2} \|U_h\|^2_{H^1(\Gamma_h(t))} + \frac{1}{\varepsilon} \int_{\Gamma_h(t)} \psi(U_h) \right) + \int_0^T \|W_h\|^2_{H^1(\Gamma_h(t))} \, dt$$
$$\leq c \left(\frac{\varepsilon}{2} \|U_{h,0}\|^2_{H^1(\Gamma_{h,0})} + \frac{1}{\varepsilon} \int_{\Gamma_{h,0}} \psi(U_h) \right). \qquad (3)$$

[3] Du, Ju, and Tian 2011

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Furthermore, there exists a constant $c > 0$ indepdent of h, t, T such that[3]

$$\|U_h\|_{L^\infty(\Gamma_h(t))} \leq c. \tag{4}$$

[3] Du, Ju, and Tian 2011

Abstract notation – continuous equations

We write

$$m(w,\varphi) = \int_{\Gamma(t)} w\varphi \qquad a(w,\varphi) = \int_{\Gamma(t)} \nabla_\Gamma w \cdot \nabla_\Gamma \varphi$$

$$g(v; w,\varphi) = \int_{\Gamma(t)} w\varphi \, \nabla_\Gamma v.$$

Abstract notation – continuous equations

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$$m(w,\varphi) = \int_{\Gamma(t)} w\varphi \qquad a(w,\varphi) = \int_{\Gamma(t)} \nabla_\Gamma w \cdot \nabla_\Gamma \varphi$$
$$g(v;w,\varphi) = \int_{\Gamma(t)} w\varphi\, \nabla_\Gamma v.$$

So the weak form becomes

$$m(\partial^\bullet u,\varphi) + g(v;u,\varphi) + a(w,\varphi) = 0$$
$$\varepsilon a(u,\varphi) + \frac{1}{\varepsilon} m(\psi'(u),\varphi) - m(w,\varphi) = 0.$$

and the variational form becomes

$$\frac{d}{dt} m(u,\varphi) + a(w,\varphi) = m(u,\partial^\bullet \varphi).$$

We write

$$m_h(W_h, \phi_h) = \int_{\Gamma_h(t)} W_h \phi_h \qquad a_h(W_h, \phi_h) = \int_{\Gamma_h(t)} \nabla_{\Gamma_h} W_h \cdot \nabla_{\Gamma_h} \phi_h$$

$$g_h(V_h; W_h, \phi_h) = \int_{\Gamma_h(t)} W_h \phi_h \, \nabla_{\Gamma_h} V_h.$$

Abstract notation - discrete equations

We write

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$$g_h(V_h; W_h, \phi_h) = \int_{\Gamma_h(t)} W_h \phi_h \, \nabla_{\Gamma_h} V_h.$$

So the weak form becomes

$$m_h(\partial_h^\bullet U_h, \phi_h) + g_h(V_h; U_h, \phi_h) + a_h(W_h, \phi_h) = 0$$

$$\varepsilon a_h(U_h, \phi_h) + \frac{1}{\varepsilon} m_h(\psi'(U_h), \phi_h) - m_h(W_h, \phi_h) = 0.$$

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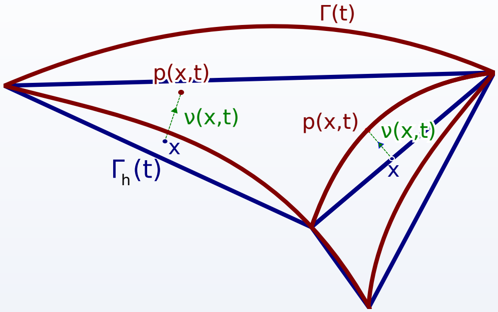
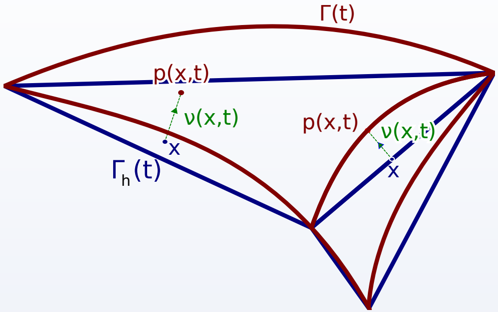

Γ(t)
p(x,t)
ν(x,t)
x
p(x,t)
ν(x,t)
x
Γh(t)

Lifted finite elements



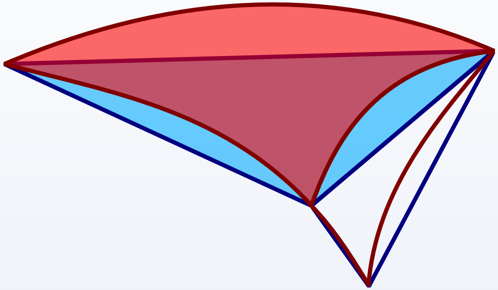
- As before, we define the lift of a finite element function using the closest point operator:

$$\phi_h^\ell(p(x,t),t) := \phi_h(x,t).$$

- We define the space of lifted finite element functions as

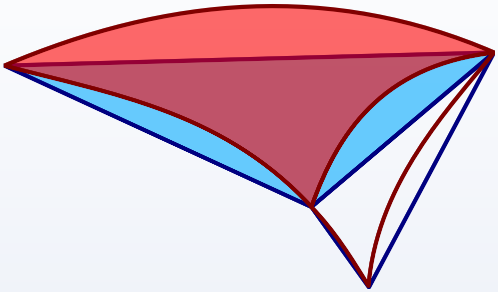
$$S_h^\ell(t) := \{\phi_h^\ell : \phi_h \in S_h(t)\}.$$

- We adopt the convention of using $\varphi_h = \phi_h^\ell$.



- This lifting process also means for each simplex $E(t)$ in $\Gamma_h(t)$, there exists a unique corresponding lifted triangle $e(t)$ in $\Gamma(t)$

$$e(t) = \{p(x,t) : x \in E(t)\}.$$



- We define another discrete material velocity on Γ which describes how the triangles $\{e(t)\}$ move. We call this v_h.
- This defines discrete material derivative on Γ for functions $\varphi_h(\cdot, t) \in S_h(t)$ such that

$$\partial_h^\bullet \varphi_h = (\partial_h^\bullet \phi_h)^\ell.$$

Lemma

For $z \in H^1(\Gamma(t))$ and $\phi_h \in S_h(t)$ with lift $\varphi_h \in S_h^\ell(t)$, we have

$$\left| m_h(z^{-\ell}, \phi_h) - m(z, \varphi_h) \right| \leq c h^2 \left\| z \right\|_{L^2(\Gamma(t))} \left\| \phi_h \right\|_{L^2(\Gamma_h(t))} \tag{5a}$$

$$\left| a_h(z^{-\ell}, \phi_h) - a(z, \varphi_h) \right| \leq c h^2 \left\| \nabla_\Gamma z \right\|_{L^2(\Gamma(t))} \left\| \nabla_{\Gamma_h} \phi_h \right\|_{L^2(\Gamma_h(t))} \tag{5b}$$

$$\left| g_h(V_h; z^{-\ell}, \phi_h) - g(v_h; z, \varphi_h) \right| \leq c h^2 \left\| z \right\|_{L^2(\Gamma(t))} \left\| \phi_h \right\|_{L^2(\Gamma_h(t))} . \tag{5c}$$

We also have bounds on the error of the different velocities:

Lemma

The difference between the continuous velocity v and the discrete velocity v_h on $\Gamma(t)$ can be estimated by

$$|v - v_h| + h\,|\nabla_\Gamma(v - v_h)| \leq c h^2. \tag{6}$$

The allows us to bound the error of the material derivatives of $\eta \in H^1(\Gamma(t))$ by

$$\|\partial^\bullet \eta - \partial_h^\bullet \eta\|_{L^2(\Gamma(t))} \leq c h^2 \|\eta\|_{H^1(\Gamma(t))} \tag{7}$$

and if $\eta \in H^2(\Gamma(t))$

$$\|\nabla_\Gamma(\partial^\bullet \eta - \partial_h^\bullet \eta)\|_{L^2(\Gamma(t))} \leq c h^2 \|\eta\|_{H^2(\Gamma(t))}. \tag{8}$$

Outline

Well posedness of the Cahn-Hilliard equation

Through showing convergence of the numerical scheme we can show the following result:

Theorem

There exists a unique solution pair (u, w) to the Cahn-Hilliard equation (1).

- We start by showing the the lifts of the finite element solutions U_h^ℓ, W_h^ℓ are bounded in $L^2(0, T; H^1(\Gamma(t))) \cap L^\infty(0, T; L^2(\Gamma(t)))$. We use Lemmas 4 and 5 to translate the results of Theorem 3.
- So we can take a weak limit along subsequences $(U_h^\ell, W_h^\ell) \rightharpoonup (\bar{u}, \bar{w})$.
- We can show that $(\bar{u}, \bar{w})$ do indeed solve the Cahn-Hilliard equation.
- Uniqueness is shown using the inverse Laplacian[4].

[4] Blowey 1990

Error Analysis – Discrete projection

For a function $z \in H^1(\Gamma(t))$ with $\int_{\Gamma(t)} z = 0$, we define the discrete projection $\Pi_h z \in S_h(t)$ of z as the unique solution of

$$a_h(\Pi_h z, \phi_h) = a(z, \varphi_h) \quad \text{for all } \phi_h \in S_h(t).$$

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Lemma

For $z \in H^2(\Gamma(t))$, we have that

$$\left\| z^{-\ell} - \Pi_h z \right\|_{L^2(\Gamma_h(t))} + h \left\| \nabla_{\Gamma_h}(z^{-\ell} - \Pi_h z) \right\|_{L^2(\Gamma_h(t))} \leq c h^2 \| z \|_{H^2(\Gamma(t))} . \tag{9}$$

Furthermore, if $\partial^\bullet z \in H^2(\Gamma(t))$, then

$$\begin{aligned} \left\| \partial_h^\bullet(z^{-\ell} - \Pi_h z) \right\|_{L^2(\Gamma_h(t))} &+ h \left\| \partial_h^\bullet \nabla_{\Gamma_h}(z^{-\ell} - \Pi_h z) \right\|_{L^2(\Gamma_h(t))} \\ &\leq c h^2 \big(\| z \|_{H^2(\Gamma(t))} + \| \partial^\bullet z \|_{H^2(\Gamma(t))} \big). \end{aligned} \tag{10}$$

We split the error into two parts using the projection from the last slide

$$\begin{aligned} u^{-\ell} - U_h &= (u^{-\ell} - \Pi_h u) + (\Pi_h u - U_h) = \rho^u + \theta^u \\ w^{-\ell} - W_h &= (w^{-\ell} - \Pi_h w) + (\Pi_h w - W_h) = \rho^w + \theta^w. \end{aligned}$$

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We already have bounds on ρ^u and ρ^w so it is left to bound θ^u and θ^w.

Simple manipulation shows

$$\begin{aligned}
&\frac{d}{dt} m_h(\theta^u, \phi_h) + a_h(\theta^w, \phi_h) - m_h(\theta^u, \partial_h^\bullet \phi_h) \\
&\quad = \big(m_h(\partial_h^\bullet \Pi_h u, \phi_h) - m(\partial_h^\bullet u, \varphi_h)\big) \\
&\qquad + \big(g_h(V_h; \Pi_h u, \phi_h) - g(v_h; u, \varphi_h)\big) \\
&\qquad + m(u, \partial^\bullet \varphi_h - \partial_h^\bullet \varphi_h).
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon a_h(\theta^u, \phi_h) + \frac{1}{\varepsilon} m_h(\psi'(\Pi_h u) - \psi'(U_h), \phi_h) - m_h(\theta^w, \phi_h) \\
&= \frac{1}{\varepsilon}\big(m_h(\psi'(\Pi_h u), \phi_h) - m(\psi'(u), \varphi_h)\big) \\
&\quad - \big(m_h(\Pi_h w, \phi_h) - m(w, \varphi_h)\big).
\end{aligned}$$

Bounds on θ^u and θ^w

Using the geometric bounds and the bounds on the discrete projection and some simple manipulation leads to

$$
\begin{aligned}
&\varepsilon \frac{d}{dt} \|\theta^u\|^2_{L^2(\Gamma_h(t))} + \|\theta^w\|^2_{L^2(\Gamma_h(t))} \\
&\quad \leq c\varepsilon \|\theta^u\|^2_{L^2(\Gamma_h(t))} \\
&\qquad + c\frac{h^4}{\varepsilon}\big(\|\partial^\bullet u\|^2_{H^2(\Gamma(t))} + \|u\|^2_{H^2(\Gamma(t))} + \|w\|^2_{H^2(\Gamma(t))} \big).
\end{aligned}
\tag{11}
$$

Combining this with the bounds on ρ^u and ρ^w gives

Theorem

Let u, w solve (1) and U_h, W_h solve (2) we have that

$$\varepsilon \sup_{t\in(0,T)} \left\| u^{-\ell} - U_h \right\|^2_{L^2(\Gamma_h(t))} + \int_0^T \left\| w^{-\ell} - W_h \right\|^2_{L^2(\Gamma_h(t))} \leq Ch^4, \tag{12}$$

with C given by

$$C = c\varepsilon \|u_0\|^2_{H^2(\Gamma_0)} + \varepsilon \sup_{t\in(0,T)} \|u\|_{H^2(\Gamma(t))} + c\frac{1}{\varepsilon} \int_0^T \left(\|\partial^\bullet u\|^2_{H^2(\Gamma(t))} + \|w\|^2_{H^2(\Gamma(t))} \right) dt.$$

- To test the convergence of our method, a linear fourth order problem ($\psi = 0$) has been implemented using the ALBERTA finite element toolbox[5].
- We used backward Euler method to do the time stepping and solved the full saddle point system directly.
- We solved on $\Gamma(t)$ given as the zero level set of the function

$$\Phi(x,t) = \frac{x_1^2}{a(t)} + x_2^2 + x_3^2 - 1 \qquad a(t) = 1 + 0.25\sin(10\pi t),$$

  with an additional right hand side f calculated so that the exact solution is given by

$$u(x,t) = \exp(-36\varepsilon t)x_1 x_2.$$

- We chose $\tau = o(h^2)$, $\varepsilon = 0.1$ and solved until $T = 0.1$.

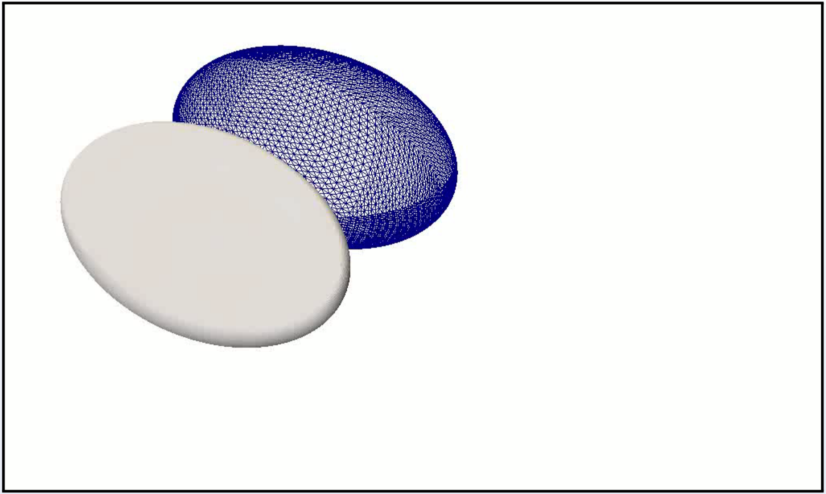
[5] Schmidt, Siebert, Köster, and Heine 2005

$u - u_h$:

h	L^2 error	eoc	H^1 error	eoc
$6.08436 \cdot 10^{-1}$	$6.3967 \cdot 10^{-2}$	-	$3.0977 \cdot 10^{-1}$	-
$3.16879 \cdot 10^{-1}$	$1.8172 \cdot 10^{-2}$	1.929092	$1.5948 \cdot 10^{-1}$	1.017693
$1.59965 \cdot 10^{-1}$	$4.6960 \cdot 10^{-3}$	1.979579	$8.0059 \cdot 10^{-2}$	1.008141
$8.01710 \cdot 10^{-2}$	$1.1850 \cdot 10^{-3}$	1.993315	$3.9968 \cdot 10^{-2}$	1.005633

$w - w_h$:

h	L^2 error	eoc	H^1 error	eoc
$6.08436 \cdot 10^{-1}$	$1.5772 \cdot 10^{-2}$	—	$2.2467 \cdot 10^{-1}$	—
$3.16879 \cdot 10^{-1}$	$4.4940 \cdot 10^{-3}$	1.924506	$9.5519 \cdot 10^{-2}$	1.311046
$1.59965 \cdot 10^{-1}$	$1.1860 \cdot 10^{-3}$	1.948837	$4.5485 \cdot 10^{-2}$	1.085402
$8.01710 \cdot 10^{-2}$	$2.9300 \cdot 10^{-4}$	2.024005	$2.2407 \cdot 10^{-2}$	1.024922



Summary

- We can use surface calculus to derive a Cahn-Hilliard equation on an evolving surface.
- We can formulate a surface finite element method to approximate solutions to this type of equation.
- Well posedness of the equations can be shown through convergence of a finite element scheme.
- We can show optimal order convergence result as well.

Challenges

- Mesh quality?
- Unknown surfaces?

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Thank you for your attention!

James F. Blowey. *The Cahn-Hilliard Gradient Theory for Phase Seperation with Non-Smooth Free Energy*. PhD thesis, University of Sussex, 1990.

Qiang Du, Lili Ju & Li Tian. Finite element approximation of the Cahn-Hilliard equation on surfaces. *Computer Methods in Applied Mechanics and Engineering*, 200:2458–2470, July 2011.

Gerhard Dziuk & Charles M. Elliott. Surface finite elements for parabolic equations. *Journal of Computational Mathemtatics*, 25 (4):385–407, 2007.

Carston Eilks & Charles M. Elliott. Numerical simulation of dealloying by surface dissolution via the evolving surface finite element method. *Journal of Computational Physics*, 227(23):9727–9741, December 2008.

Alfred Schmidt, Kunibert G. Siebert, Daniel Köster & Claus-Justus Heine. *Design of adaptive finite element software: the finite element toolbox ALBERTA*. Springer-Verlag, Berlin-Heidelberg, 2005.