Optimal closed-loop controls via finite system of control devices for reaction-diffusion processes

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Problem statement: general idea

- A physical process evolves over a bounded domain.
- A given number of control devices and measurement devices is at our disposal

How to place the devices to keep the process possibly close to a given reference state?

Example: an open loop control for a semilinear problem

$$\begin{cases} y_t - \Delta y = f(y) + \hat{u} & \text{on } \Omega \times (0, T) \\ \frac{\partial y}{\partial n} = 0 & \text{on } \partial\Omega \times (0, T) \\ y(0, x) = y_0(x) & \text{for } x \in \Omega \end{cases}$$

where $\hat{u}$ — the control term, $\Omega \subset \mathbb{R}^d$ — a bounded domain.

The aim: to keep the evolution of the process as close as possible to a given target state $y^*(x)$, $x \in \Omega$.

A problem: Let $y^* \equiv 0$. Then y^* is an unstable equilibrium for the nonlinear term given by:

$$f(s) = s - s^3$$

Adding the control devices

$$y_t(x,t) - \Delta y(x,t) = f(y(x,t)) + \sum_{j=1}^{J} g_j(x)\kappa_j(t)$$

where

g_j	—	functions describing the control devices, $j = 1, \ldots, J$
κ_j	—	functions describing the actions of the devices;
		these are not prescribed functions,
		we assume that κ_j depends on a solution y itself

Describing the dependence of κ_j on the solution complements the model.

Complementing the feedback law

$$\begin{cases} \kappa_j'(t) + \kappa_j(t) = W_j(y(\,.\,,t), y^*(\,.\,,t)) & \text{on } [0,T] \\ \kappa_j(0) = \kappa_{j0} \in \mathbb{R} & \text{for } j = 1,\dots,J \end{cases}$$

and

$$W_j(y, y^*) = \sum_{k=1}^{K} \alpha_{jk} w_k \left(\int_\Omega h_k(y - y^*)dx \right)$$

where

h_k	—	functions describing the measurement devices,
w_k	—	functions describing a data processing algorithm, e.g. $w_k = -sgn$,
α_{jk}	—	nonnegative weights, for every j we have $\sum_{k=1}^{K} \alpha_{jk} = 1$.
y^*	—	the reference state (or trajectory), $y^* = y^*(x,t)$

References

- Hoffmann, K.-H., Niezgódka, M., and Sprekels, J.
  Nonlinear Anal.: Theory, Meth. and Appl. 15 (1990), 955–976.
- Dudziuk, G., and Niezgódka, M.
  Adv. Math. Sci. Appl. 21, 2 (2011), 383–402.

Main system of equations — the control system

$$\begin{cases} y_t - \Delta y = f(y) + \sum_{j=1}^{J} \hat{u}_{g_j}(x)\kappa_j(t) & \text{on } \Omega \times (0,T) \\ \kappa_j'(t) + \kappa_j(t) = \sum_{k=1}^{K} \hat{u}_{\alpha_{jk}} w_k \left(\int_\Omega \hat{u}_{h_k}(y - y^*)dx \right) & \text{on } [0,T] \\ & \text{for } j = 1, \dots, J \\ \frac{\partial y}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T) \\ y(0,x) = y_0(x) & \text{for } x \in \Omega \\ \kappa_j(0) = \kappa_{j0} \in \mathbb{R} & \text{for } j = 1, \dots, J \end{cases}$$

Where we call the following sequence *a control*

$$\hat{u} = (\hat{u}_{g_1}, \dots, \hat{u}_{g_J}, \hat{u}_{h_1}, \dots, \hat{u}_{h_k}, \hat{u}_{\alpha_{11}}, \dots, \hat{u}_{\alpha_{JK}})$$

Let us denote the *state operator* as $S = (S_y, S_{\kappa_1}, \dots, S_{\kappa_J})$:

$$S : \hat{u} \longmapsto (y, \kappa_1, \dots, \kappa_J) =: (S_y(\hat{u}), S_{\kappa_1}(\hat{u}), \dots, S_{\kappa_J}(\hat{u}))$$

Control space and the set of admissible controls

The space of controls $\hat{u}$ will be denoted by U. We call U *a control space* and consider its two variants:

$$U = U^0 = \left(L^2(\Omega)\right)^J \times \left(L^2(\Omega)\right)^K \times \mathbb{R}^{KJ}$$
$$U = U^1 = \left(H^1(\Omega)\right)^J \times \left(H^1(\Omega)\right)^K \times \mathbb{R}^{KJ}$$

Weights $\hat{u}_{\alpha_{jk}}$ should be nonnegative and summable to 1, thus we define the *set of admissible controls* as:

$$U_{ad} = \left\{ \hat{u} \in U : \sum_{k=1}^{K} \hat{u}_{\alpha_{jk}} = 1 \ \forall_j \text{ and } \hat{u}_{\alpha_{jk}} \geq 0 \ \forall_{j,k} \right\}$$

Existence results

Basic assumptions

- $\Omega \subset \mathbb{R}^d$ is a bounded domain (of sufficiently smooth boundary),
- Nonlinear terms f and w_k, $k = 1, \ldots, K$ are Lipschitz continuous,
- $y_0 \in L^2(\Omega)$ and U is one of U^0 or U^1.

Theorem 1

Under the basic assumptions above, the weak solution to the control system exists and is unique in the space

$$\begin{aligned} X \quad = \quad & \{y \in L^\infty(0, T; L^2(\Omega)),\ \nabla y \in L^2(\Omega \times (0, T)), \\ & y' \in L^2(0, T; H^1(\Omega)^*) \text{ and} \\ & \kappa_j \in L^\infty(0, T),\ \kappa_j' \in L^2(0, T) \text{ for } j = 1, \ldots, J\} \end{aligned}$$

As a consequence, the state operator S is well defined from U into X.

The experiment

We present results of simulations for the main control system (performed with use of Octave).

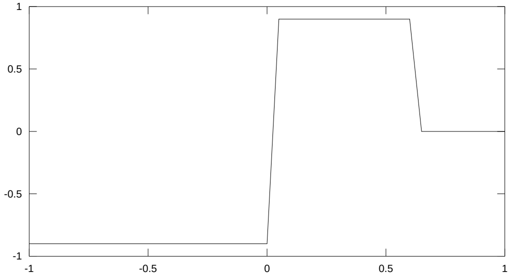
The experiment assumptions:

- We take $\Omega = (-1, 1) \subset \mathbb{R}$ and $T = 4$,
- We assume $f(s) = s - s^3$,
- We assume linearity of w_k, namely $w_k(s) = -50s$,
- We assume $y^* \equiv 0$,
- We assume that control devices are simply characteristic functions of disjoint intervals of the same lenght covering the domain,
- We assume the same for the measurement devices and put $K = J$ — in consequence, every measurement device covers one of the control devices.

The experiment

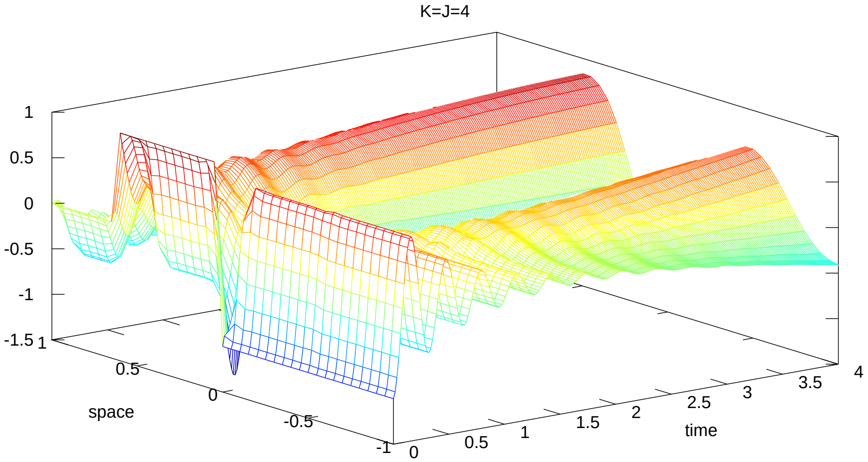
Moreover:

- We take an „arbitrary" initial condition:

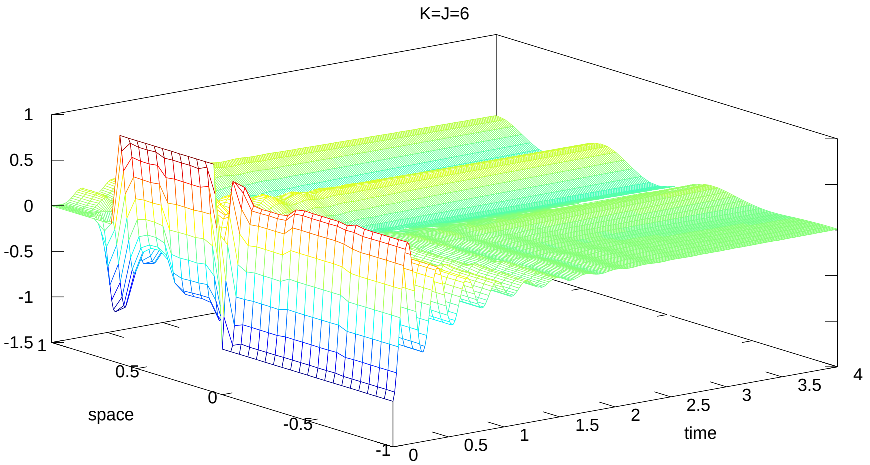


We have executed our experiment with also with other initial conditions bounded by 1 and the results were similar as the ones on the following slides.

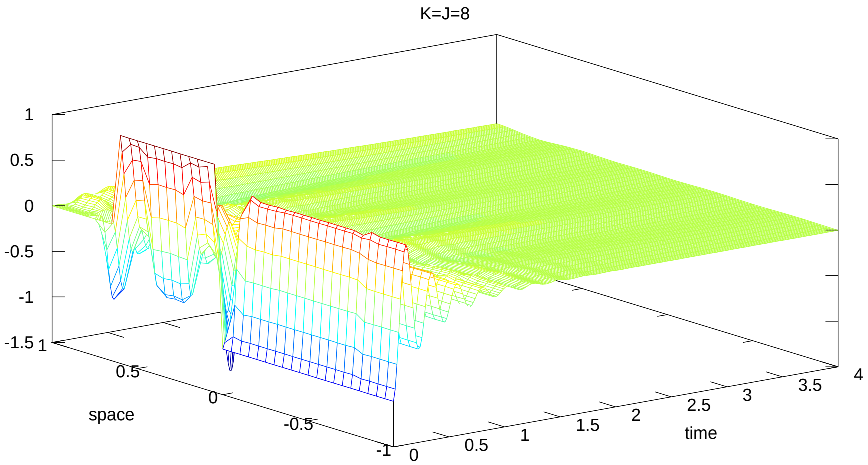
4 control devices, 4 measurement units



6 control devices, 6 measurement units



8 control devices, 8 measurement units



The optimality criterion

The cost functional is defined as:

$$\mathcal{J}_\lambda(\hat{u}) = \| S_y(\hat{u}) - y^* \|^2_{L^2(\Omega\times(t_1,T))} + \lambda \, | \hat{u} |^2_U$$

where y^*, $t_1 \in [0, T)$ and $\lambda \geq 0$ are given and the seminorm $| \hat{u} |_U$ on U is defined by:

$$| \hat{u} |_U = \| (\hat{u}_{g_1}, \ldots, \hat{u}_{g_J}, \hat{u}_{h_1}, \ldots, \hat{u}_{h_k}, 0, \ldots, 0) \|_U$$

Problem statement: precise formulation

For given choice of the control space U, find $\hat{u} \in U_{ad}$ solving the problem

$$\inf_{\hat{u}\in U_{ad}} \mathcal{J}_\lambda(\hat{u})$$

Stability results

Theorem 2

Under the basic assumptions above, the state operator

$$S\colon U^0 \longrightarrow X$$

is Lipschitz continuous on bounded subsets of U^0.

Theorem 3

Under the basic assumptions above, the state operator

$$S\colon U^0_{weak} \longrightarrow X_{weak}$$

is a closed operator.

Solvability of the opimization problem

Theorem 4

Assume that

- the basic assumptions set holds true,
- $\lambda > 0$,

then the optimization problem has at least one solution.

Idea of the proof:

- $\lambda > 0$, hence the minimizing sequence is bounded
- for $U = U^0$ we extract a weakly convergent subsequence and use the weak continuity of S for the limit passage
- for $U = U^1$ we extract a strongly convergent subsequence in U^0 and use the strong continuity of S for the limit passage

Differentiability of $\mathcal{J}_\lambda$

Theorem 5

Assume that

- the basic assumptions set holds true,
- and moreover the nonlinear terms f and w_k are everywhere differentiable,
- $\lambda \geq 0$,
- $U = U^0$,

then the state operator S is weakly Gâteaux differentiable.

Fact: The square of norm in the Hilbert space H is Fréchet differentiable
Fact: A superposition of a weakly Gâteaux differentiable operator with a Fréchet differentiable functional is Gâteaux differentiable.
Conclusion: $\mathcal{J}_\lambda$ is Gâteaux differentiable.

The adjoint system

We define the *adjoint system in point $\hat{u}$* (as before, $y = S_y(\hat{u})$):

$$\begin{cases} -\widetilde{p}_t - \Delta\widetilde{p} - f'(y)\widetilde{p} = (y - y^*)\mathbf{1}_{(t_1,T)} + \\ \qquad + \sum_{j=1}^{J} \sum_{k=1}^{K} \hat{u}_{\alpha_{jk}} w_k' \left(\int_\Omega \hat{u}_{h_k}(y - y^*)\, dx \right) \hat{u}_{h_k} \widetilde{q}_j & \text{on } Q_T \\ -\widetilde{q}_1' + \widetilde{q}_1 = \int_\Omega \hat{u}_{g_1} \widetilde{p}\, dx & \text{on } [0, T] \\ \vdots & \vdots \\ -\widetilde{q}_J' + \widetilde{q}_J = \int_\Omega \hat{u}_{g_J} \widetilde{p}\, dx & \text{on } [0, T] \\ \\ \frac{\partial \widetilde{p}}{\partial n} = 0 \qquad \text{on } \partial\Omega \times (0, T) \\ \widetilde{p}(T, x) \equiv 0 \\ \widetilde{q}_j(T) = 0 \quad \forall_{j=1,\dots,J} \end{cases}$$

This system will be usefull for characterization of $D_G \mathcal{J}_\lambda(\bar{u})(\hat{v})$.

Necessary optimality conditions - the main theorem

Fix $\hat{u} \in U = U^0$ and denote

- $(y, \kappa_1, \ldots, \kappa_J)$ as a solution of the main control system corresponding to $\hat{u}$, i.e. $(y, \kappa_1, \ldots, \kappa_J) = S(\hat{u})$,
- $(\widetilde{p}, \widetilde{q}_1, \ldots, \widetilde{q}_J)$ as a solution of the adjoint system corresponding to $\hat{u}$.

Theorem 6

Let the assumptions as in Theorem 5 be fulfilled. Then the Gâteaux differential of $\mathcal{J}_\lambda$ in $\hat{u}$ is given by

$$(D_G \mathcal{J}_\lambda)(\hat{u})(\hat{v}) = (\hat{f}, \hat{v})_U \qquad \forall_{\hat{v} \in U}$$

where the element $\hat{f} \in U^* = U$ is defined as:

$$\begin{aligned}
\hat{f}_{g_j} &= 2 \int_0^T \widetilde{p} \kappa_j \, dt + 2\lambda \hat{u}_{g_j} \\
\hat{f}_{h_k} &= 2 \int_0^T \hat{u}_{\alpha_{jk}} w_k' \left(\int_\Omega \hat{u}_{h_k} (y - y^*) \, dx \right) (y - y^*) \, \widetilde{q}_j \, dt + 2\lambda \hat{u}_{h_k} \\
\hat{f}_{\alpha_{jk}} &= 2 \int_0^T w_k \left(\int_\Omega \hat{u}_{h_k} (y - y^*) \, dx \right) \widetilde{q}_j \, dt
\end{aligned}$$

Remarks

Remark 1: We have expressed the differential $(D_G\mathcal{J}_\lambda)(\hat{u})(\,.\,)$ in terms of the main control system and the adjoint system.

Remark 2: We can use it for concluding the necessary optimality criterion: if $\hat{u}$ is optimal in U_{ad} w.r.t. our cost functional, then

$$(\hat{f}, \hat{w} - \hat{u})_U \geq 0 \qquad \forall_{\hat{w}\in U_{ad}}$$

Remark 3: Formula for $\hat{f}$ is in fact a formula for the gradient of $\mathcal{J}_\lambda$ in point $\hat{u}$ and it can be utilized for the implementation of the gradient methods of optimization, applicable for numerical searching of optimal elements.

Thank you for attention

The presented content will be a part of Grzegorz Dudziuk's Ph.D. thesis, supervised by Marek Niezgdka (ICM, Warsaw University).