Reversing interfaces in non-linear diffusion processes with absorption

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Outline

Problem description: the general case

Concerned with compactly supported solutions to

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h^m \frac{\partial h}{\partial x}\right) - h^{1-q},$$

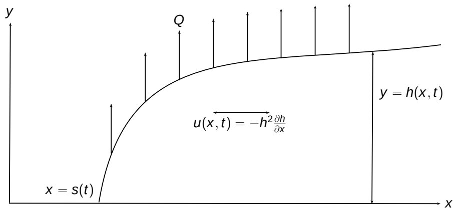
with $m > 0$, $q > 0$ and $m - q > 0$. Initial and boundary conditions are

$$h = h_0(x) \quad \text{when} \quad t = -\tau,$$

$$h = 0 \quad \text{and} \quad \frac{ds}{dt} = -h^{m-1}\frac{\partial h}{\partial x} + \left(q \frac{\partial}{\partial x}(h^q)\right)^{-1} \quad \text{at} \quad x = s(t)$$

plus some analogous conditions at the right interface.

A gravity driven viscous film with evaporation, $m = 3$ and $q = 1$



A numerical solution (with appropriate initial conditions) will typically

- Spread out due to gravity
- The spreading slows down as the fluid evaporates
- Begins to recede
- Becomes extinct

Travelling wave solutions of the PDE local to a left interface

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right) - 1.$$

An advancing wave (controlled by diffusion)

$$h \sim \left(-3\frac{ds}{dt}\right)^{1/3} (x - s(t))^{1/3}$$

and a receding wave (controlled by absorption)

$$h \sim \left(\frac{ds}{dt}\right)^{-1} (x - s(t)).$$

But, how does an advancing wave become a receding wave?

A local self-similar solution

Look for a local self-similar solution. For $t < 0$ (prior to the reversing time) we write

$$h = (-t)H(\phi), \quad \phi = x(-t)^{-2} \text{ and } s(t) = A(-t)^2,$$

and for $t > 0$ (after the reversing time) we write

$$h = tH(\phi), \ \phi = xt^{-2} \text{ and } s(t) = Bt^2.$$

Match the two parts of the solution together at $t = 0$ by insisting that the far field behaviours of H are the same.

Asymptotics for the ODEs

For $t < 0$: Behaviour near the interface

$$H \sim (6A)^{1/3}(\phi - A)^{1/3} \quad \text{as} \quad \phi \to A^+.$$

For $t < 0$: Behaviour in the far field

$$H \sim N\phi^{1/2} \quad \text{as} \quad \phi \to +\infty.$$

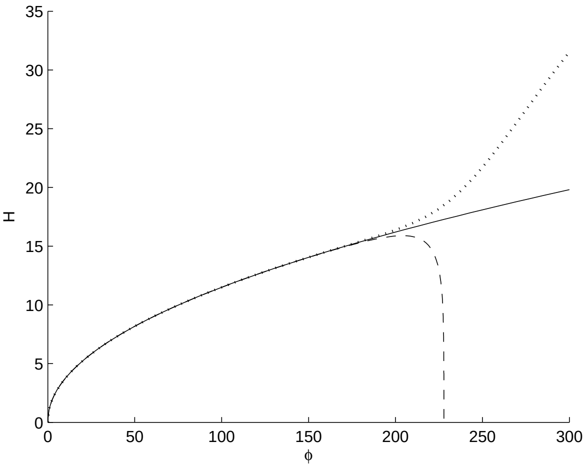
For $t > 0$: Behaviour near the interface

$$H \sim (2B)^{-1}(\phi - B) \quad \text{as} \quad \phi \to B^+.$$

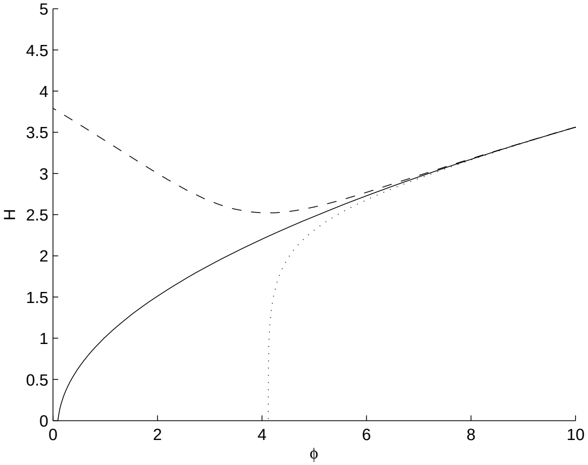
For $t > 0$: Behaviour in the far field

$$H \sim Q\phi^{1/2} \quad \text{as} \quad \phi \to +\infty.$$

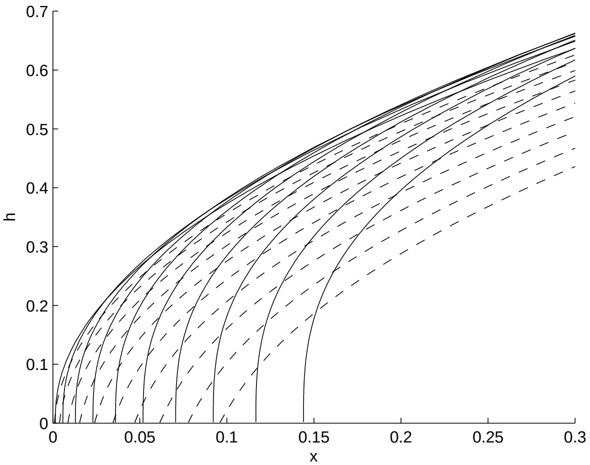
Prior to the reversing time



After the reversing time



Local solution to the PDE

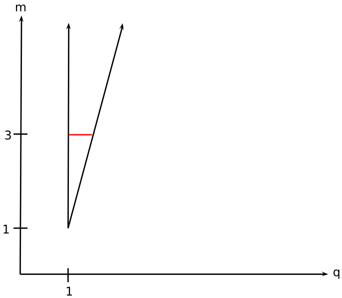


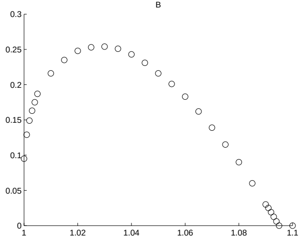
Back to the general case

Can this be done for any pairs of values of m and q?

- For all values of m and q that satisfy $m > 0$, $q > 0$ and $m - q > 0$ there are suitable self-similar reductions.
- There are also plausible asymptotic behaviours of the resulting ODEs.
- Everything works nicely for $q = 1$ (and any value of $m > 1$).
- For $q < 1$ (absorption $\propto h^{1-q}$) can only find a solution with $A = 0$, and, cannot match to a solution for $t > 0$. Conjecture that the similarity solution breaks down.

- For $q > 1$ (absorption $\propto h^{1-q}$) can get some solutions, but, only in a rather limited range.
- Outside this range $B = 0$, and, cannot match to a solution for $t < 0$. Conjecture that the similarity solution breaks down.





- Why does everything go wrong for $q < 1$? Is the absorption too weak?
- Why is there such a limited range of values $q > 1$ that work? Is the absorption too strong?
- For any (working) pairs of values of m and q the solution is uniquely determined by a local analysis. Does this mean that the reversing behaviour we have found is generic? Or are there some other types of interface reversal that are driven by global effects?
- Are there other families of equations that could be analysed using these techniques?

Reference: J. M. Foster et. al. *The reversing of interfaces in slow diffusion processes with strong absorption.* SIAM Appl. Math. 72(1):144-162, 2012.