The longest shortest fence and sharp Poincaré-Sobolev inequalities*

Vincenzo Ferone

Università degli Studi di Napoli Federico II

12. International Conference on

Free Boundary Problems

Theory and Applications

June 11-15, 2012
CHIEMSEE

*Joint work with L. Esposito, B. Kawohl, C. Nitsch and C. Trombetti, ARMA (to appear)

Problem 1

In the class of planar convex sets having fixed area, which set maximizes the length of the shortest area-bisecting arc?

G. PÓLYA. *Aufgabe 283.* Elem. d. Math. **13** (1958), 40–41.
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Convex Body Isoperimetric Conjecture

Conjecture

The least perimeter to enclose given volume inside an open ball in $\mathbb{R}^n$ is greater than inside any other convex body of the same volume.

Problem 2

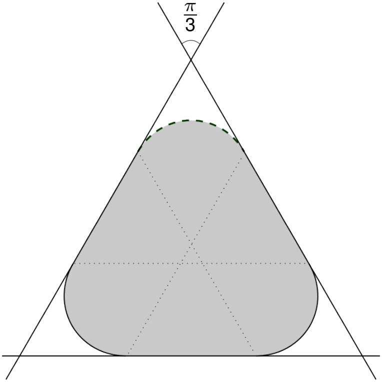
In the class of convex sets having fixed area, which set maximizes the length of the shortest bisecting chord?

H. AUERBACH. *Sur un problème de M. Ulam concernant l'équilibre des corps flottants* Studia Math. **7** (1938), 121–142.

Auerbach triangle

$$\begin{cases} x(t) = \dfrac{e^{4t}-1}{e^{4t}+1} - t \\ \\ y(t) = 2\dfrac{e^{2t}}{e^{4t}+1} \end{cases}$$

$$t \in [-(\log 3)/4, (\log 3)/4]$$



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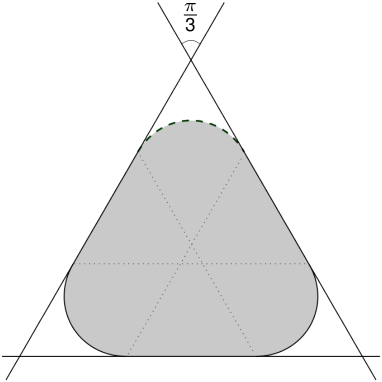

All bisecting chords of Auerbach triangle have constant length which is bigger than the diameter of the circle with the same area.

Theorem (Problem 1)

The disc solves Problem 1 (it has the longest shortest area-bisecting arc).

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Theorem (Problem 2)

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Length of the shortest bisecting arc

$$\inf_{\substack{G \subset K \\ |G| = \frac{|K|}{2}}} Per(G; K)$$

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Theorem (Problem 1)

If K is an open convex set of $\mathbb{R}^2$, we have:

$$\inf_{\substack{G \subset K \\ |G| = \frac{|K|}{2}}} Per(G; K)^2 \leq \frac{4}{\pi}|K|.$$

Moreover, equality holds above if and only if K is a disc.

Problem 1

Length of the shortest bisecting arc

$$\inf_{\substack{G\subset K \\ |G|=\frac{|K|}{2}}} Per(G;K)$$

Theorem (Problem 1)

If K is an open convex set of $\mathbb{R}^2$, we have:

$$\inf_{\substack{G\subset K \\ |G|=\frac{|K|}{2}}} \frac{Per(G;K)^2}{|G|} \leq \frac{8}{\pi}.$$

Moreover, equality holds above if and only if K is a disc.

A remark about the centrosymmetric case

Pólya observed that, if K is a convex centrosymmetric set, then an upper bound on the length L of the shortest bisecting curve is given by

$$L \leq 2\sqrt{|K|/\pi} \tag{1}$$

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If K is centrosymmetric (with respect to the origin), $x \in K$ implies $-x \in K$.

If $\bar{x} \in \partial K$, then the chord delimited by $\bar{x}$ and $-\bar{x}$ bisects K. Inequality (1) follows from the fact that there exists $\bar{x} \in \partial K$ such that

$$|\bar{x}| \leq \sqrt{|K|/\pi}.$$

Indeed, the disc centered at the origin and having same area as K cannot be strictly contained in K.

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The result holds true in the whole class of measurable centrosymmetric sets.

However the restriction to the centrosymmetric case is somewhat misleading since one may believe that working with chords instead of curves could be sufficient to answer Problem 1.

Relative isoperimetric inequality in $\mathbb{R}^2$

Relative isoperimetric constant ($\alpha \geq 1/2$)

$$\gamma_\alpha(K) = \inf_{\substack{G \subset K \\ 0<|G|<|K|}} \frac{Per(G;K)}{(\min\{|G|, |K \setminus G|\})^\alpha}.$$

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Theorem

If K is a convex set in $\mathbb{R}^2$ and $\alpha \geq 1/2$, we have

$$\gamma_\alpha(K) \leq \gamma_\alpha(K^\sharp), \tag{2}$$

where $K^\sharp$ is the disc such that $|K^\sharp| = |K|$. Equality holds if and only if K is a disc.

A Sobolev-Poincaré inequality in $\mathbb{R}^2$

Sobolev-Poincaré inequality

$$\|Du\|(K) \geq I(K)\|u - \bar{u}\|_2, \qquad u \in BV(K),$$

where $\|Du\|(K)$ is the total variation of u in K and $\bar{u}$ is the mean value of u on K.

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$$I(K) = |K|^{1/2} \inf_{\substack{G \subset K \\ 0<|G|<|K|}} \frac{Per(G;K)}{\sqrt{|G|\,|K \setminus G|}}.$$

A. Cianchi (1989)

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A. CIANCHI (1989)

Problem

For which set $I(K)$ attains its biggest value?

M. ZHU (2004), H. BREZIS - J. VAN SCHAFTINGEN (2008)

A Sobolev-Poincaré inequality in $\mathbb{R}^2$

Theorem

$$I(K) \leq I(K^\sharp)$$

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Proof

We observe that

$$I(K) = |K|^{1/2} \inf_{\substack{G \subset K \\ 0<|G|<|K|}} \frac{Per(G;K)}{\sqrt{|G|\,|K \setminus G|}} \leq |K|^{1/2}\gamma_1(K) = \sqrt{2}\gamma_{1/2}(K),$$

and that

$$I(K^\sharp) = \sqrt{2}\gamma_{1/2}(K^\sharp).$$

Neumann eigenvalue in $\mathbb{R}^2$

H. Gajewski (2001)

$$\mu_1(K) = \inf_{\substack{u \in BV(K) \\ u \neq 0,\ \int_K \text{sign } u = 0}} \frac{\|Du\|(K)}{\|u\|_1}$$

first non-trivial Neumann eigenvalue of the p-Laplacian with $p \to 1$.

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Theorem (Szegö-Weinberger inequality)

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Theorem (Szegö-Weinberger inequality)

$$\mu_1(K) \leq \mu_1(K^\sharp)$$

(Observe that $\mu_1(K) = \gamma_1(K)$)

Definition

We say that a convex set K is a set with constant halving length (CHL-set) if each point of its boundary is a terminal point of a bisecting curve with minimal length (optimal arc).

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Strategy of the proof (Problem 1):

- There exists a set which solves Problem 1 (a consequence of Blaschke selection theorem)
- Every set which solves Problem 1 is a CHL-set
- Among CHL-sets with fixed measure the disc has the minimal bisecting curve of maximal length

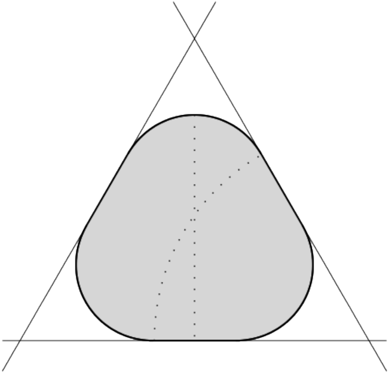
Definition

We say that a convex set K is a Zindler set if each point of its boundary is a terminal point of a bisecting chord with minimal length.

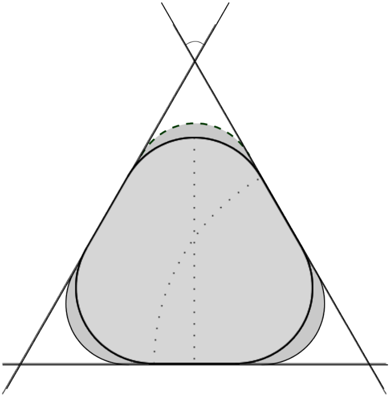
Strategy of the proof (Problem 2):

- There exists a set which solves Problem 2 (a consequence of Blaschke selection theorem)
- Every set which solves Problem 2 is a Zindler set
- Among Zindler sets with fixed measure Auerbach triangle has the minimal bisecting chord of maximal length [N. Fusco - A. Pratelli (2010)]

CHL triangle



CHL triangle and Auerbach triangle



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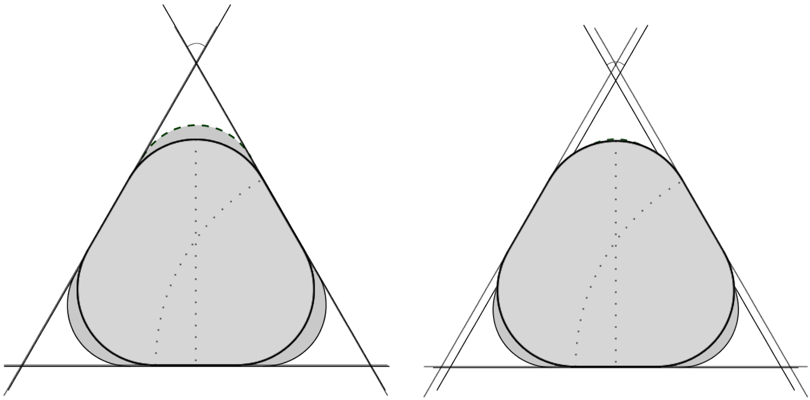
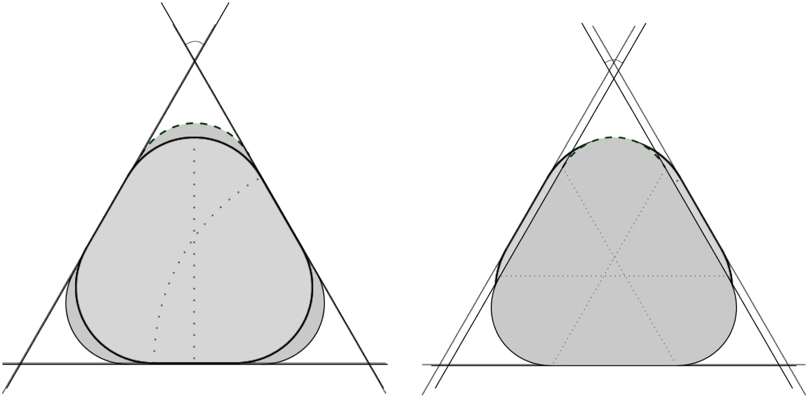


Area of CHL triangle $\simeq 0.7981\ldots$ $(L = 1)$

Area of Auerbach triangle $\simeq 0.7755\ldots$

$(\pi/4 \simeq 0.7854\ldots)$

CHL triangle and Auerbach triangle



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Let K be an open convex set of $\mathbb{R}^2$. There exists a convex set of measure $\dfrac{|K|}{2}$, which minimizes

$$\inf_{\substack{G \subset K \\ 0<|G|\leq \frac{|K|}{2}}} \frac{Per(G;K)^2}{|G|}$$

and any minimizer E has the following properties

(a) $\partial E \cap K$ is either a circular arc or a straight segment.
The opening angle is at most $\sqrt{3}$.

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(c) $\partial E \cap K$ is orthogonal to ∂K.

(d) If $|E| < \dfrac{|K|}{2}$, then E is a circular sector having sides on ∂K.

Proposition

Let K be an open convex set of $\mathbb{R}^2$. There exists a half-plane H which minimizes

$$\inf_{\substack{F\subset\mathbb{R}^2 \text{ half-plane} \\ 0<|F\cap K|\leq\frac{|K|}{2}}} \frac{Per(F\cap K;K)^2}{|F\cap K|}$$

such that $|H\cap K|=|K|/2$, and any minimizer H has the following properties

(a) *Let P be one of the terminal points of $\partial H\cap K$. Then P is a regular point of ∂K in the sense that ∂K has a tangent line at P.*

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(b) *The tangent lines to ∂K at the two terminal points of $\partial H\cap K$ either bound with $\partial H\cap K$ an isosceles triangle or they are orthogonal to $\partial H\cap K$.*

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(a) *Let P be one of the terminal points of $\partial H \cap K$. Then P is a regular point of ∂K in the sense that ∂K has a tangent line at P.*

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(c) *If $|H\cap K| < |K|/2$, then $H \cap K$ is an isosceles triangle having sides on ∂K.*

Optimal arc

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and $|E| = \frac{|K|}{2}$

Reduction to a CHL-set

Lemma

If K^* solves Problem 1, then K^* is a CHL-set.

Reduction to a CHL-set

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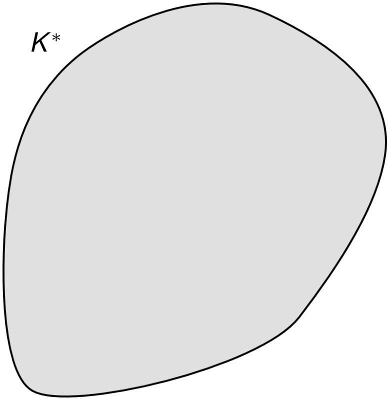
If K^* solves Problem 1, then K^* is a CHL-set.

The strategy of the proof consists in showing that, if K^* is not a CHL-set, then it is always possible to "cut off a piece of K^*" in such a way that

$$\mathcal{C}(K) = \inf_{\substack{G \subset K \\ 0 < |G| \leq \frac{|K|}{2}}} \frac{Per(G; K)^2}{|G|}$$

strictly increases.

Reduction to a CHL-set



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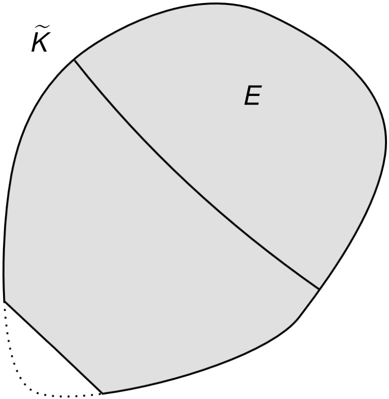
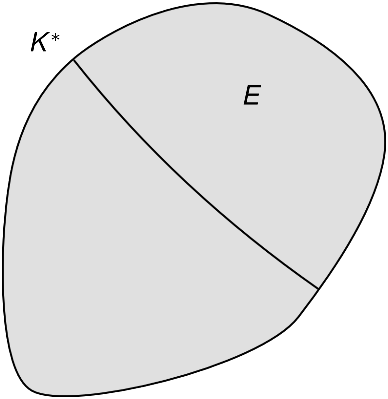


$$\mathcal{C}(\widetilde{K}) = \frac{Per(E; \widetilde{K})^2}{|E|}$$

Reduction to a CHL-set



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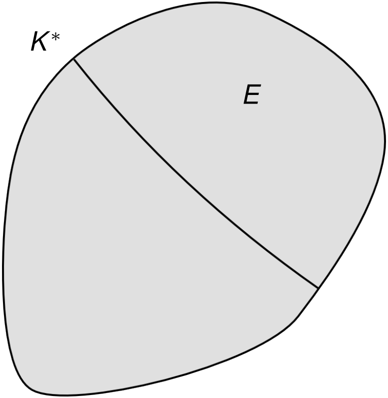
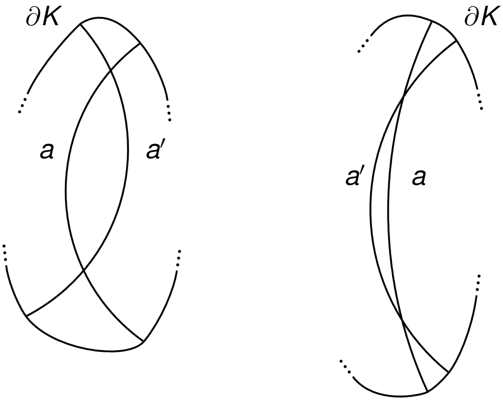


$$\mathcal{C}(\widetilde{K}) = \frac{Per(E;\widetilde{K})^2}{|E|} = \frac{Per(E;K^*)^2}{|E|} \geq \inf_{\substack{G\subset K^* \\ 0<|G|\leq \frac{|K^*|}{2}}} \frac{Per(G;K^*)^2}{|G|} = \mathcal{C}(K^*)$$

Two intersections

Lemma

Two optimal arcs cross each other transversally in one and only one point.



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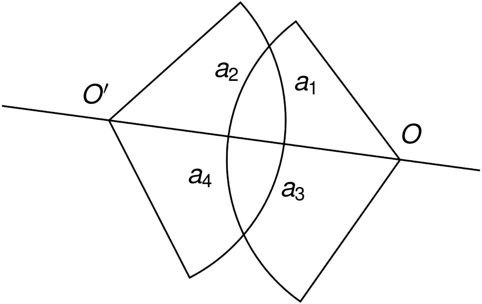

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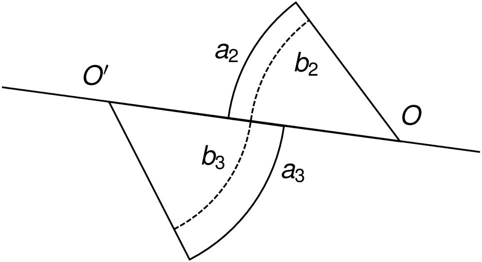

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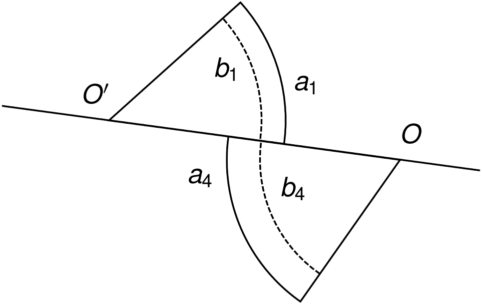

Optimization in the class of CHL-sets

Aim

In the class of CHL-sets with given measure we look for the set which has the optimal arcs of maximal length.

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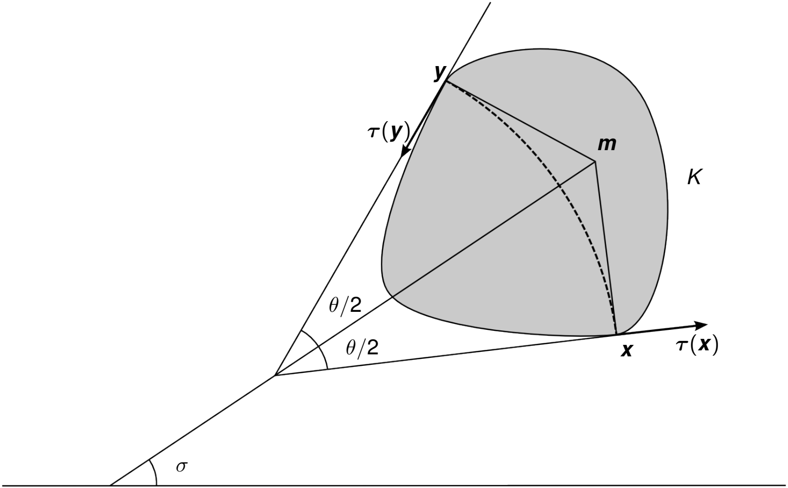
$$\Updownarrow$$

Aim

In the class of CHL-sets with given length L of the bisecting arc we look for the set which has the minimal measure.

CHL-sets: regularity and parametric representation



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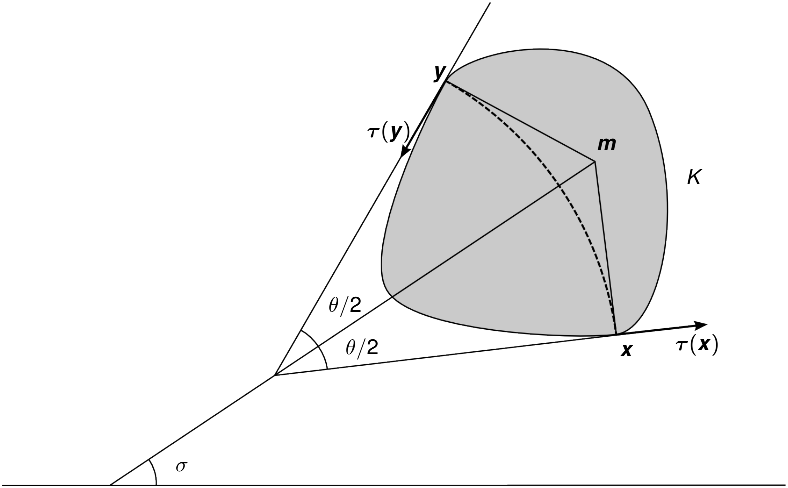

Lemma

Any CHL-set is of class $C^{1,1}$ and $\mathbf{x}(\sigma)$, $\mathbf{y}(\sigma)$, $\theta(\sigma)$, $\mathbf{m}(\sigma)$ are lipschitz functions.

Parametric representation of a CHL-set

If $g(\tau) = \dfrac{L}{\tau} \tan \dfrac{\tau}{2}$ (g is extended by continuity in $\tau = 0$), we have

$$\boldsymbol{x}(\sigma) = \boldsymbol{m}(\sigma) - g(\theta(\sigma))(-\sin(\sigma - \theta(\sigma)/2), \cos(\sigma - \theta(\sigma)/2))$$

$$\boldsymbol{y}(\sigma) = \boldsymbol{m}(\sigma) + g(\theta(\sigma))(-\sin(\sigma + \theta(\sigma)/2), \cos(\sigma + \theta(\sigma)/2))$$

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Differentiating $\boldsymbol{x}$ and $\boldsymbol{y}$ and using the fact that the optimal arc touches the boundary of K orthogonally, we have:

$$\boldsymbol{m}'(\sigma) \cdot (-\sin\sigma, \cos\sigma) = 0.$$

This means that, for every σ, the vector $\boldsymbol{m}'(\sigma)$ points in the direction $(\cos\sigma, \sin\sigma)$, then

$$\boldsymbol{m}'(\sigma) = M(\sigma)(\cos\sigma, \sin\sigma).$$

Parametric representation of a CHL-set

Furthermore, using the fact that

$$\boldsymbol{x}'(\sigma) = \left[M(\sigma) \sin \frac{\theta(\sigma)}{2} - \frac{d}{d\sigma}(g(\theta(\sigma))) \right] (-\sin(\sigma - \theta(\sigma)/2), \cos(\sigma - \theta(\sigma)/2))$$
$$+ \left[M(\sigma) \cos \frac{\theta(\sigma)}{2} + g(\theta(\sigma)) \left(1 - \frac{\theta'(\sigma)}{2} \right) \right] (\cos(\sigma - \theta(\sigma)/2), \sin(\sigma - \theta(\sigma)/2))$$

$$\boldsymbol{y}'(\sigma) = \left[-M(\sigma) \sin \frac{\theta(\sigma)}{2} + \frac{d}{d\sigma}(g(\theta(\sigma))) \right] (-\sin(\sigma + \theta(\sigma)/2), \cos(\sigma + \theta(\sigma)/2))$$
$$+ \left[M(\sigma) \cos \frac{\theta(\sigma)}{2} - g(\theta(\sigma)) \left(1 + \frac{\theta'(\sigma)}{2} \right) \right] (\cos(\sigma + \theta(\sigma)/2), \sin(\sigma + \theta(\sigma)/2)),$$

Parametric representation of a CHL-set

Furthermore, using the fact that

$$\boldsymbol{x}'(\sigma) = \left[M(\sigma)\sin\frac{\theta(\sigma)}{2} - \frac{d}{d\sigma}(g(\theta(\sigma)))\right](-\sin(\sigma-\theta(\sigma)/2), \cos(\sigma-\theta(\sigma)/2))$$
$$+\left[M(\sigma)\cos\frac{\theta(\sigma)}{2} + g(\theta(\sigma))\left(1-\frac{\theta'(\sigma)}{2}\right)\right](\cos(\sigma-\theta(\sigma)/2), \sin(\sigma-\theta(\sigma)/2))$$

$$\boldsymbol{y}'(\sigma) = \left[-M(\sigma)\sin\frac{\theta(\sigma)}{2} + \frac{d}{d\sigma}(g(\theta(\sigma)))\right](-\sin(\sigma+\theta(\sigma)/2), \cos(\sigma+\theta(\sigma)/2))$$
$$+\left[M(\sigma)\cos\frac{\theta(\sigma)}{2} - g(\theta(\sigma))\left(1+\frac{\theta'(\sigma)}{2}\right)\right](\cos(\sigma+\theta(\sigma)/2), \sin(\sigma+\theta(\sigma)/2)),$$

we have that M and θ are related by the following equality

$$M(\sigma) = \frac{1}{\sin\frac{\theta(\sigma)}{2}}\frac{d}{d\sigma}(g(\theta(\sigma)))$$

Parametric representation of a CHL-set

Lemma

For every CHL-set there exists a lipschitz function $\theta(\sigma)$, $\sigma \in [-\pi, \pi[$, *which satisfies*

$$\theta(\sigma - \pi) = -\theta(\sigma), \qquad \forall \sigma \in [0, \pi[, \tag{3}$$

and such that

$$\boldsymbol{m}'(\sigma) = M(\sigma)(\cos\sigma, \sin\sigma)$$

$$\boldsymbol{x}'(\sigma) = \left[M(\sigma)\cos\frac{\theta(\sigma)}{2} + g(\theta(\sigma))\left(1 - \frac{\theta'(\sigma)}{2}\right)\right]\mathbf{e}_-(\sigma)$$

$$\boldsymbol{y}'(\sigma) = \left[M(\sigma)\cos\frac{\theta(\sigma)}{2} - g(\theta(\sigma))\left(1 + \frac{\theta'(\sigma)}{2}\right)\right]\mathbf{e}_+(\sigma),$$

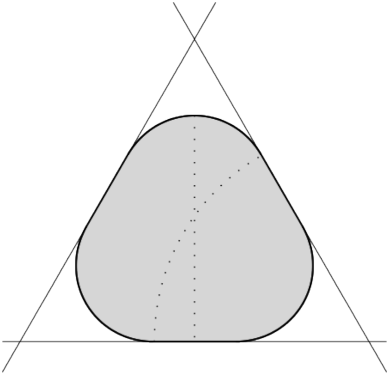
where

$$\mathbf{e}_\pm(\sigma) = (\cos(\sigma \pm \theta(\sigma)/2), \sin(\sigma \pm \theta(\sigma)/2)).$$

CHL triangle

Choosing $|\theta'(\sigma)| = 2$

$$\theta(\sigma) = \frac{\pi - \big|2\pi - |6\sigma - 3\pi|\big|}{3}, \qquad \sigma \in [0, \pi[,$$



Measure of a CHL-set K

By Gauss-Green formula we have:

$$|K| = \frac{1}{2}\int_0^{\pi} (\boldsymbol{x}(\sigma) \wedge \boldsymbol{x}'(\sigma) + \boldsymbol{y}(\sigma) \wedge \boldsymbol{y}'(\sigma))\, d\sigma$$

$$= \int_0^{\pi} \int_0^{t} M(t)M(\sigma)\,\sin(t-\sigma)\, d\sigma\, dt + \int_0^{\pi} g^2(\theta(\sigma))\, d\sigma.$$

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Putting

$$f(\tau) = \frac{L}{2}\int_0^{\tau} \frac{1}{t\sin(t/2)}\left(1 - \frac{2}{t}\tan\frac{t}{2} + \tan^2\frac{t}{2}\right)\,dt,$$

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Putting

$$f(\tau) = \frac{L}{2}\int_0^\tau \frac{1}{t\sin(t/2)}\left(1 - \frac{2}{t}\tan\frac{t}{2} + \tan^2\frac{t}{2}\right)\,dt,$$

that is

$$\frac{d}{dt}f(\theta(t)) = M(t).$$

and integrating by parts...

Measure of a CHL-set K

By Gauss-Green formula we have:

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$$= \int_0^{\pi}\int_0^{t} f(\theta(t))f(\theta(\sigma))\, \sin(t-\sigma)\, d\sigma\, dt - \int_0^{\pi} f^2(\theta(\sigma))\, d\sigma + \int_0^{\pi} g^2(\theta(\sigma))\, d\sigma.$$

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$$= \int_0^{\pi} \int_0^{t} f(\theta(t))f(\theta(\sigma)) \sin(t-\sigma)\, d\sigma\, dt - \int_0^{\pi} f^2(\theta(\sigma))\, d\sigma + \int_0^{\pi} g^2(\theta(\sigma))\, d\sigma.$$

Lemma

$$\int_0^{\pi} \int_0^{t} f(\theta(t))f(\theta(\sigma)) \sin(t-\sigma)\, d\sigma\, dt + \frac{1}{8}\int_0^{\pi} f^2(\theta(\sigma))\, d\sigma \geq 0.$$

Measure of a CHL-set K

$$|K| \geq -\frac{9}{8}\int_0^{\pi} f^2(\theta(\sigma))\,d\sigma + \int_0^{\pi} g^2(\theta(\sigma))\,d\sigma$$

Measure of a CHL-set K

$$|K| \geq -\frac{9}{8}\int_0^{\pi} f^2(\theta(\sigma))\,d\sigma + \int_0^{\pi} g^2(\theta(\sigma))\,d\sigma$$

Being

$$g^2(\tau) - \frac{9}{8}f^2(\tau) \geq \frac{L^2}{4}, \qquad \tau \in [-\sqrt{3}, \sqrt{3}],$$

we have

$$|K| \geq \frac{\pi}{4}L^2$$