$C^{1,1}$ regularity for degenerate elliptic obstacle problems in mathematical finance

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A degenerate-elliptic obstacle problem I

We consider questions of existence, uniqueness, and regularity of solutions, $u : \mathscr{O} \to \mathbb{R}$, to the obstacle problem,

$$\begin{aligned} \min\{Au - f, u - \psi\} &= 0 \quad \text{on } \mathscr{O}, \\ u &= g \quad \text{on } \Gamma_1, \end{aligned}$$

given a

- Possibly unbounded domain, $\mathscr{O} \subset \mathbb{H} := \mathbb{R} \times \mathbb{R}_+$, with $\mathbb{R}_+ = (0, \infty)$;
- Boundary portion, $\Gamma_1 = \partial\mathscr{O} \cap \mathbb{H}$, transverse to $\partial\mathbb{H}$;
- Source function, $f : \mathscr{O} \to \mathbb{R}$;
- Dirichlet boundary data function, $g : \mathscr{O} \cup \Gamma_1 \to \mathbb{R}$;
- An obstacle function, $\psi : \mathscr{O} \cup \Gamma_1 \to \mathbb{R}$, which is compatible with the boundary data, $\psi \leq g$ on Γ_1;

A degenerate-elliptic obstacle problem II

Here, A is the degenerate-elliptic Heston operator,

$$Av := -\frac{y}{2}\left(v_{xx} + 2\rho\sigma v_{xy} + \sigma^2 v_{yy}\right) - (r - q - y/2)v_x - \kappa(\theta - y)v_y + rv, \quad v \in C^\infty(\mathbb{H}).$$

The (constant) coefficients of A obey the ellipticity conditions

$$\sigma \neq 0 \quad \text{and} \quad -1 < \rho < 1,$$

and $\kappa > 0$, $\theta > 0$, $q \geq 0$, and $r \geq 0$.

The operator, $-A$, is the generator of the Heston stochastic volatility process with killing, a degenerate diffusion process often used to model asset prices in mathematical finance.

While A is an elliptic differential operator on $\mathscr{O}$, it becomes degenerate along $\Gamma_0 := \text{int}(\partial\mathbb{H} \cap \partial\mathscr{O})$, where $y = 0$.

A degenerate-elliptic obstacle problem III

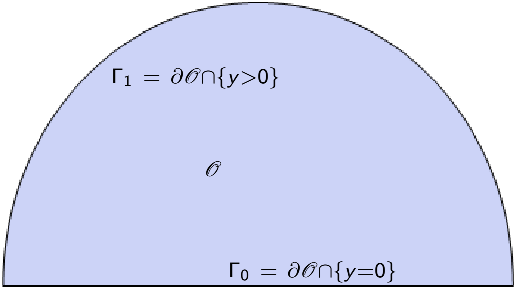

Figure: Boundaries for degenerate elliptic problems on subdomains of the half-space.

A degenerate-elliptic obstacle problem IV

Because $\kappa\theta > 0$, no boundary condition needs to be prescribed along Γ_0. Instead, the degenerate elliptic boundary problem,

$$Au = f \quad \text{a.e. on } \mathscr{O}, \quad u = g \quad \text{on } \Gamma_1,$$

is well-posed when we seek solutions in

- Weighted Sobolev spaces, $H^1(\mathscr{O}, \mathfrak{w})$ (variational inequality) or $H^2(\mathscr{O}, \mathfrak{w})$ (strong);
- Weighted Hölder spaces, $C_s^{2+\alpha}(\mathscr{O} \cup \Gamma_0) \cap C(\bar{\mathscr{O}})$ or $C_s^{2+\alpha}(\bar{\mathscr{O}})$ (classical);

These function spaces select solutions, u, from two possible families, C^∞ or C^0 up to Γ_0.

Motivation from option pricing in mathematical finance

A solution u to the

- Elliptic obstacle problem when $f = 0$ can be interpreted as the value function for a perpetual American-style option with payoff function given by the obstacle function, ψ
- Parabolic obstacle problem on $\mathscr{O} \times [0, T]$, with $0 < T < \infty$, can be interpreted as the value function for a finite-maturity American-style option with payoff function given by a terminal condition function, $h : \mathscr{O} \to \mathbb{R}$, which typically coincides on $\mathscr{O} \times \{T\}$ with the obstacle function, ψ.

For the American-style put option, when $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+$,

$$\psi(x, y) = (E - e^x)^+, \quad (x, y) \in \mathbb{H},$$

where $E > 0$ is a positive constant, x is the log-price of a financial asset, and y is an internal variable (variance).

Numerical solution of the degenerate elliptic obstacle problem

- The theoretical methods of existence of solutions to the variational equations and inequalities defined by the Heston operator, including the special attention to types of boundary conditions described here, may be implemented using finite element methods, while solutions to the corresponding boundary value or obstacle problems may be implemented using finite difference methods.
- The numerical solution of these degenerate elliptic and parabolic problems is the Ph.D. thesis topic of Eduardo Osorio (May 2013) and graphs of these solutions, in the elliptic case, are illustrated in the following slides, with $\psi(x, y) = (E - e^x)^+$ in the case of the obstacle problem.

Numerical solution to the boundary value problem

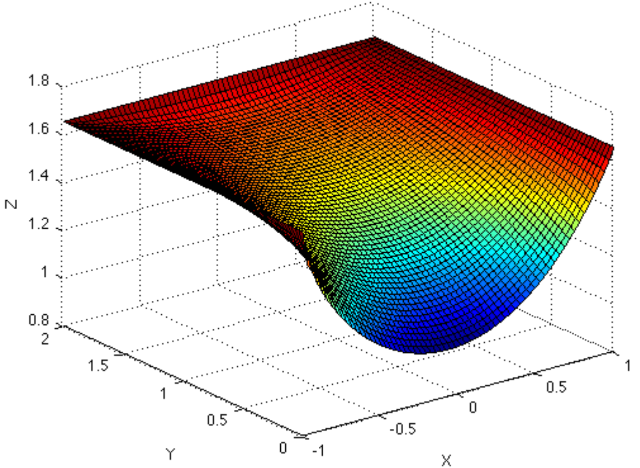

Figure: Numerical solution to the degenerate elliptic boundary value problem, $Au = f$ on $\mathscr{O}$ and $u = g$ on $\Gamma_1 = \partial\mathscr{O} \cap \{y > 0\}$.

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Numerical solution to the obstacle problem

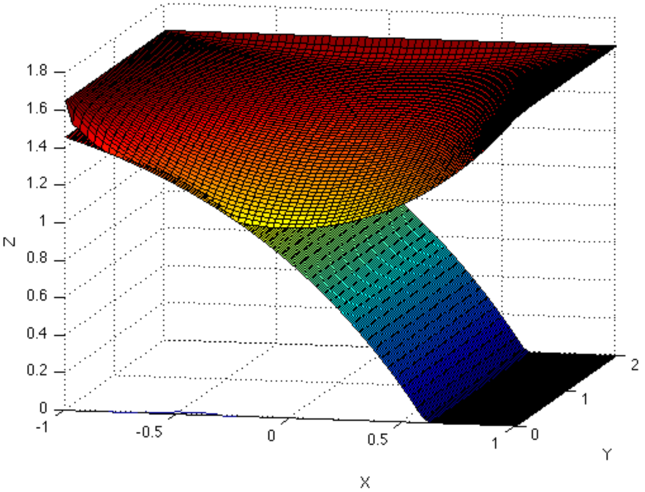

Figure: Numerical solution to the degenerate elliptic obstacle problem, $\min\{Au - f, u - \psi\} = 0$ a.e. on $\mathscr{O}$ and $u = g$ on $\Gamma_1 = \partial\mathscr{O} \cap \{y > 0\}$.

Weighted L^2, H^1, and H^2 Sobolev spaces

We need a weight function when defining our Sobolev spaces,

$$\mathfrak{w}(x,y) := y^{\beta-1} e^{-\gamma|x|-\mu y}, \quad \beta = \frac{2\kappa\theta}{\sigma^2}, \quad \mu = \frac{2\kappa}{\sigma^2},$$

for $(x,y) \in \mathbb{H}$ and a suitable positive constant, γ. We define

$$L^2(\mathscr{O},\mathfrak{w}) := \left\{ u \in L_{\text{loc}}(\mathscr{O}) : u\,\mathfrak{w}^{1/2} \in L^2(\mathscr{O}) \right\},$$

$$H^1(\mathscr{O},\mathfrak{w}) := \left\{ u \in L_{\text{loc}}(\mathscr{O}) : (1+y)^{1/2}u,\ y^{1/2}Du \in L^2(\mathscr{O},\mathfrak{w}) \right\},$$

$$H^2(\mathscr{O},\mathfrak{w}) := \left\{ u \in L_{\text{loc}}(\mathscr{O}) : (1+y)^{1/2}u,\ (1+y)Du,\ yD^2u \in L^2(\mathscr{O},\mathfrak{w}) \right\}$$

where $Du = (u_x, u_y)$ and $D^2u = (u_{xx}, u_{xy}, u_{yx}, u_{yy})$ are defined in the sense of distributions.

Weighted L^2, H^1, and H^2 Sobolev norms

The spaces $H^1(\mathscr{O},\mathfrak{w})$, $H^1(\mathscr{O},\mathfrak{w})$, $H^1(\mathscr{O},\mathfrak{w})$ are Hilbert spaces with respect to the norms defined by

$$\|u\|^2_{L^2(\mathscr{O},\mathfrak{w})} := \int_{\mathscr{O}} u^2\,\mathfrak{w}\,dx\,dy,$$

$$\|u\|^2_{H^1(\mathscr{O},\mathfrak{w})} := \int_{\mathscr{O}} \left(y|Du|^2 + 1 + y)u^2\right)\,\mathfrak{w}\,dx\,dy,$$

$$\|u\|^2_{H^2(\mathscr{O},\mathfrak{w})} := \int_{\mathscr{O}} \left(y^2|D^2u|^2 + (1+y)^2|Du|^2 + (1+y)u^2\right)\,\mathfrak{w}\,dx\,dy.$$

Recall that $C_0^\infty(\mathscr{O}\cup\Gamma_0)$ is the subspace of functions $u\in C^\infty(\mathscr{O})$ such that $u\in C^\infty(\bar{U})$, for every $U \Subset \mathscr{O}\cup\Gamma_0$.

Define $H_0^1(\mathscr{O}\cup\Gamma_0,\mathfrak{w})$ to be the closure in $H^1(\mathscr{O},\mathfrak{w})$ of $C_0^\infty(\mathscr{O}\cup\Gamma_0)$.

Existence and uniqueness of strong solutions to the obstacle problem

Theorem (Existence and uniqueness of strong solutions to the obstacle problem)

Assume $\mathscr{O} \subset \mathbb{H}$ *is bounded, that* Γ_2 *is* C^2*-transverse to* Γ_0*, and that* $r > 0$*, where* r *is a coefficient of* A*. Let* $f \in L^2(\mathscr{O}, \mathfrak{w})$ *and* $\psi \in H^2(\mathscr{O}, \mathfrak{w})$ *such that* $\psi^+ \in L^\infty(\mathscr{O}) \cap H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$*. Then there is a unique solution* $u \in H^2(\mathscr{O}, \mathfrak{w}) \cap H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$ *to*

$$\min\{Au - f, u - \psi\} = 0 \quad \textit{a.e. on } \mathscr{O},$$

and u *obeys*

$$\begin{aligned}\|u\|_{H^1(\mathscr{O},\mathfrak{w})} &\leq C_1 \left(\|u\|_{L^2(\mathscr{O},\mathfrak{w})} + \|(f\|_{L^2(\mathscr{O},\mathfrak{w})} + \|\psi^+\|_{H^1(\mathscr{O},\mathfrak{w})}\right), \\ \|u\|_{H^2(\mathscr{O},\mathfrak{w})} &\leq C_2 \left(\|u\|_{L^2(\mathscr{O},\mathfrak{w})} + \|(f\|_{L^2(\mathscr{O},\mathfrak{w})} + \|\psi\|_{H^2(\mathscr{O},\mathfrak{w})}\right),\end{aligned}$$

for some positive constants, $C_i = C_i(A, \mathscr{O})$*.*

Additional (but suboptimal) regularity

Theorem (Hölder continuity)

Assume the hypotheses required for existence of solutions in $H^1_0(\mathscr{O}\cup\Gamma_0,\mathfrak{w})$ to the variational inequality formulation of the obstacle problem. If in addition $f\in L^q(\bar{\mathscr{O}},\mathfrak{w})$ for $q>2+\beta$ and $u\in H^1_0(\mathscr{O}\cup\Gamma_0,\mathfrak{w})$ is a solution to the variational inequality, then $u\in C^{\alpha_0}(\bar{\mathscr{O}})$ for some $\alpha_0\in[0,1)$.

Theorem ($W^{2,p}$ regularity in interior and up to Γ_1)

Assume the hypotheses required for existence of solutions in $H^2(\mathscr{O},\mathfrak{w})$ of solutions to the obstacle problem. If $u\in H^2(\mathscr{O},\mathfrak{w})\cap H^1_0(\mathscr{O}\cup\Gamma_0,\mathfrak{w})$ is a solution and, for $2<p<\infty$,

$$f\in L^p_{\mathrm{loc}}(\mathscr{O}\cup\Gamma_1)\quad\text{and}\quad\psi\in W^{2,p}_{\mathrm{loc}}(\mathscr{O}\cup\Gamma_1),$$

and the boundary portion Γ_1 is $C^{2+\alpha}$. Then $u\in W^{2,p}_{\mathrm{loc}}(\mathscr{O}\cup\Gamma_1)$ and, if $\alpha=1-2/p$, then $u\in C^{1,\alpha}(\mathscr{O}\cup\Gamma_1)$.

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Optimal regularity for non-degenerate obstacle problems

Suppose, temporarily, that A is now a strictly elliptic, linear, second-order differential operator on $\mathscr{O} \Subset \mathbb{R}^d$,

$$Au := -a^{ij}u_{x_ix_j} - b^i u_{x_i} + cu, \quad u \in C^\infty(\mathscr{O}).$$

Then a result of Jensen (1980) gives

Theorem (Optimal regularity for solutions to *non-degenerate* elliptic obstacle problems)

Assume $\mathscr{O} \Subset \mathbb{R}^d$ is a domain of class $C^{3,1}$ and that the coefficients of A obey $a^{ij} \in C^{0,1}(\bar{\mathscr{O}})$ and $b^i, c \in C^\alpha(\bar{\mathscr{O}})$, for $0 < \alpha < 1$. Suppose $f \in C^\alpha(\bar{\mathscr{O}})$, and $g \in C^{2+\alpha}(\bar{\mathscr{O}})$, and $\psi \in C^{1,1}(\bar{\mathscr{O}})$. If $u \in H^2(\mathscr{O})$ solves

$$\min\{Au - f, u - \psi\} = 0 \quad a.e.\ on\ \mathscr{O}, \quad u - g \in H^1_0(\mathscr{O}),$$

then $u \in C^{1,1}(\bar{\mathscr{O}})$.

Optimal regularity up to the non-degenerate boundary

For the Heston operator, A, we have an easy corollary:

Corollary ($C^{1,1}$ regularity up to *non-degenerate* boundary, Γ_1)

Assume $\mathscr{O} \Subset \mathbb{R}^d$ is a domain with $C^{3,1}$ boundary portion, Γ_1, which is $C^{3,1}$-transverse to Γ_0. Suppose $f \in C^\alpha(\mathscr{O} \cup \Gamma_1)$, and $g \in C^{2+\alpha}(\mathscr{O} \cup \Gamma_1)$, and $\psi \in C^{1,1}(\mathscr{O} \cup \Gamma_1)$. If $u \in H^2(\mathscr{O}, \mathfrak{w})$ solves

$$\min\{Au - f, u - \psi\} = 0 \quad \text{a.e. on } \mathscr{O}, \quad u - g \in H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w}),$$

then $u \in C^{1,1}(\mathscr{O} \cup \Gamma_1)$.

For regularity of the solution up to the degenerate boundary, Γ_0, we shall need certain weighted Hölder spaces first defined by Daskalopoulos and Hamilton (1998) and Koch (1999).

Hölder norms defined by the cycloidal metric I

Definition ($C_s^{1,1}$ norm and Banach space)

We say that $u \in C_s^{1,1}(\bar{\mathscr{O}})$ if u belongs to $C^{1,1}(\mathscr{O}) \cap C^1(\bar{\mathscr{O}})$ and

$$\|u\|_{C_s^{1,1}(\bar{\mathscr{O}})} := \|yD^2u\|_{L^\infty(\mathscr{O})} + \|Du\|_{C(\bar{\mathscr{O}})} + \|u\|_{C(\bar{\mathscr{O}})} < \infty.$$

Also, we say that $u \in C_s^{1,1}(\mathscr{O} \cup \Gamma_0)$, if $u \in C_s^{1,1}(\bar{U})$ for any subdomain $U \Subset \mathscr{O} \cup \Gamma_0$.

A cycloidal distance function, equivalent to that of the cycloidal metric, $y^{-1}(dx^2 + dy^2)$ on $\mathbb{H}$, due to Daskalopoulos and Hamilton (1998) and Koch (1999), is given by,

$$s(z, z_0) := \frac{|x - x_0| + |y - y_0|}{\sqrt{y} + \sqrt{y_0} + \sqrt{|x - x_0| + |y - y_0|}},$$

for all $z = (x, y), z_0 = (x_0, y_0) \in \mathbb{H}$. This is the natural metric for our degenerate equation.

Hölder norms defined by the cycloidal metric II

The following weighted Hölder spaces were introduced by Daskalopoulos and Hamilton (1998).

Definition (C_s^α and $C_s^{2+\alpha}$ norms and Banach spaces)

Given $\alpha \in (0,1)$, we say that $u \in C_s^\alpha(\bar{\mathscr{O}})$ if $u \in C(\bar{\mathscr{O}})$ and

$$\|u\|_{C_s^\alpha(\bar{\mathscr{O}})} := \|u\|_{C(\bar{\mathscr{O}})} + \sup_{\substack{z_1, z_2 \in \mathscr{O} \\ z_1 \neq z_2}} \frac{|u(z_1) - u(z_2)|}{s(z_1, z_2)^\alpha} < \infty.$$

We say that $u \in C_s^{2+\alpha}(\bar{\mathscr{O}})$ if $u,\ Du,\ yD^2 u \in C_s^\alpha(\bar{\mathscr{O}})$. We denote

$$\|u\|_{C_s^{2+\alpha}(\bar{\mathscr{O}})} := \|u\|_{C_s^\alpha(\bar{\mathscr{O}})} + \|Du\|_{C_s^\alpha(\bar{\mathscr{O}})} + \|yD^2 u\|_{C_s^\alpha(\bar{\mathscr{O}})}.$$

Hölder norms defined by the cycloidal metric III

- One can show that $C^{1,1}_s(\bar{\mathscr{O}})$, $C^{\alpha}_s(\bar{\mathscr{O}})$, and $C^{2+\alpha}_s(\bar{\mathscr{O}})$ are Banach spaces when equipped with the indicated norms.
- We say that $u \in C^{\alpha}_s(\mathscr{O} \cup \Gamma_0)$ if $u \in C^{\alpha}_s(\bar{U})$ for every subdomain $U \Subset \mathscr{O} \cup \Gamma_0$ and similarly that $u \in C^{2+\alpha}_s(\mathscr{O} \cup \Gamma_0)$ if $u \in C^{2+\alpha}_s(\bar{U})$ for every subdomain $U \Subset \mathscr{O} \cup \Gamma_0$.
- One can show that if $u \in C^{2+\alpha}_s(\mathscr{O} \cup \Gamma_0)$, then

$$yu_{xx} = yu_{xy} = yu_{yy} = 0 \quad \text{on } \Gamma_0,$$

which may be viewed as a type of Ventcel (1959) or second-order boundary condition (see, for example, Taira (2004)).

Optimal regularity up to the degenerate boundary

Theorem ($C^{1,1}$ regularity up to degenerate boundary)

Suppose that $u \in H^2(\mathscr{O}, \mathfrak{w}) \cap C(\bar{\mathscr{O}})$ is a solution to

$$\min\{Au - f, u - \psi\} = 0 \quad \textit{a.e. on } \mathscr{O},$$

given $f \in C^\alpha_s(\bar{\mathscr{O}})$ and $\psi \in C^{2+\alpha}(\bar{\mathscr{O}})$. Then, $u \in C^{1,1}_s(\mathscr{O} \cup \Gamma_0)$ and, for each precompact subdomain $\mathscr{O}' \Subset \mathscr{O} \cup \Gamma_0$, there is a constant C, depending on $\alpha, \mathscr{O}', \mathscr{O}$, and the coefficients of the operator A, such that

$$\|u\|_{C^{1,1}_s(\bar{\mathscr{O}}')} \leq C \left(\|u\|_{C(\bar{\mathscr{O}})} + \|f\|_{C^\alpha_s(\bar{\mathscr{O}})} + \|\psi\|_{C^{1,1}(\bar{\mathscr{O}})} \right).$$

Work in progress I

1. **Degenerate parabolic obstacle problem.** We expect that our results on existence, uniqueness, and regularity of solutions to the degenerate elliptic obstacle problem should extend without difficulty to the corresponding parabolic problem.
2. **Regularity of the free boundary.** Previous results on the regularity of the free boundary for the obstacle problem defined by a non-degenerate elliptic or parabolic operator extend to degenerate operators of the kind considered in this article. (See the forthcoming book by Petrosyan, Shagholian, and Ural'tseva (2012) and references therein for the non-degenerate elliptic case and articles by Laurence and Salsa (2009), Nyström (2007) and references therein for the non-degenerate parabolic case.)

Work in progress II

3. **Lipschitz obstacle function.** The $C^{2+\alpha}(\bar{\mathscr{O}})$ regularity property assumed for the obstacle function, ψ, in the statement of our $C^{1,1}$ regularity theorem does not reflect the more typical Lipschitz regularity for ψ encountered in applications to mathematical finance, such as $\psi(x,y) = \max\{E - e^x, 0\}$, where E is a positive constant, in the case of a put option. Nevertheless, simple one-dimensional examples, results of Laurence and Salsa (2009), Nyström (2007), and numerical analysis due to Osorio (2012) suggest that the solution, u, should nevertheless have the optimal $C^{1,1}_s$ regularity even when $\psi = \max\{E - e^x, 0\}$.

Work in progress III

4. **Degenerate elliptic and parabolic obstacle problems with variable coefficients in higher dimensions.** We expect that our results on existence, uniqueness, and regularity of solutions to the degenerate elliptic obstacle problem on $\mathscr{O} \subseteqq \mathbb{R} \times \mathbb{R}_+$ may be easily generalized to higher dimensions, $d \geq 2$, and degenerate elliptic operators on $\mathscr{O} \subseteqq \mathbb{R}^{d-1} \times \mathbb{R}_+$ with variable coefficients,

$$Au = -x_d a_{ij} u_{x_i x_j} - b_i u_{x_i} + cu,$$

under the assumptions that (a_{ij}) is strictly elliptic, $b_d \geq \nu > 0$, for some constant $\nu > 0$, and $c \geq 0$ and all coefficients are Hölder continuous of class $C^\alpha_s(\bar{\mathscr{O}})$, for some $\alpha \in (0,1)$

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THANK YOU FOR YOUR ATTENTION!