Existence and regularity of mean curvature flow via Allen-Cahn equation

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0. Introduction

One-parameter family of smooth k-dimensional surfaces with no boundary $\{M_t\}_{t\geq 0}$ is MCF if

v (velocity) = H (mean curvature vector)

For $\phi \in C_c^1(\mathbb{R}^n;\mathbb{R}^+)$

$$\frac{d}{dt}\Big(\int_{M_t}\phi\, d\mathcal{H}^k\Big) = \int_{M_t}(-H\phi+\nabla\phi)\cdot v\, d\mathcal{H}^k$$

$$= \int_{M_t}(-H\phi+\nabla\phi)\cdot H\, d\mathcal{H}^k.$$

$\leq$ is sufficient for $v = H$

$\phi\equiv 1$ ⇨ $\frac{d}{dt}(\mathcal{H}^k(M_t)) \leq -\int_{M_t}|H|^2\, d\mathcal{H}^k \leq 0$

1. Brakke's MCF (`78)

– Existence of generalized MCF

<u>Definition</u>. A family of k-varifolds $\{V_t\}_{t\geq 0}$ in $\mathbb{R}^n$ is called Brakke's MCF if

$$\int_{\mathbb{R}^n} \phi \, d\|V_t\| \Big|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} dt \int_{\mathbb{R}^n} (-H\phi + \nabla\phi) \cdot H \, d\|V_t\|$$

for $0 \leq \forall t_1 < \forall t_2 < \infty, \ \forall \phi \in C_c^1(\mathbb{R}^n; \mathbb{R}^+)$.

$$\int_{\mathbb{R}^n} \phi \, d\|V_t\| \approx \int_{M_t} \phi \, d\mathcal{H}^k$$

H : generalized mean curvature vector of V_t

Theorem (Brakke) For a given integral k-varifold V_0 there exists a family of k-varifold $\{V_t\}_{t\geq 0}$ which is Brakke's MCF and V_t is integral for a.e. t.

Definition. V is integral k-varifold if

$$\int_{\mathbb{R}^n} \phi \, d\|V\| = \int_M \phi(x)\theta(x)\, d\mathcal{H}^k(x)$$

for $\forall \phi \in C_c(\mathbb{R}^n)$.

M : countably k-rectifiable set Ex. $\cup_{i=1}^{\infty}$ (k-dim C^1manifold M_i), and any measurable subset of such.

θ : a.e. integer-valued multiplicity function

If θ is 1 a.e., we say V is of unit density.

Relevance to the well-known existence result:

Let $\psi(x,t)$ be the unique viscosity solution of

$$\frac{\psi_t}{|\nabla\psi|} = \mathrm{div}\left(\frac{\nabla\psi}{|\nabla\psi|}\right) \quad \text{on} \quad \mathbb{R}^n \times (0,\infty)$$

$$\psi(x,0) = \text{signed distance function of } M_0.$$

For $c \in \mathbb{R}$ define $M_t^c := \{\psi(\cdot,t) = c\}$

If everything is smooth, $\{M_t^c\}_{t\geq 0}$ is MCF.

(Evans-Spruck '93) For a.e. $c \in \mathbb{R}$, $\{M_t^c\}_{t\geq 0}$ is Brakke's MCF of unit density.

$$\int_{\mathbb{R}^n} \phi \, d\|V_t\| := \int_{M_t^c} \phi \, d\mathcal{H}^{n-1} \qquad \int_{\mathbb{R}^n} \phi \, d\|V_t\| \Big|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} dt \int_{\mathbb{R}^n} (-H\phi + \nabla\phi) \cdot H \, d\|V_t\|$$

The proof uses compensated compactness type argument.

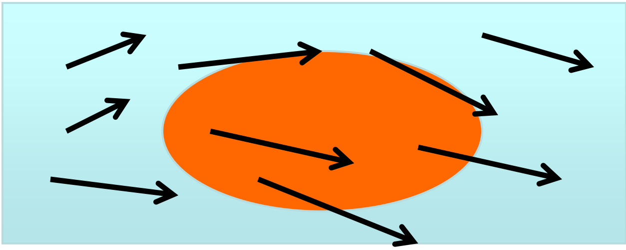
The similar results for anisotropic MCF (T. `00).

2. Existence and regularity of MCF + given u

Problem. Given a (not so regular) vector field

$$u = u(x,t) : \mathbb{T}^n \times (0,\infty) \to \mathbb{R}^n$$

and a hypersurface M_0 find a family of hypersurfaces $\{M_t\}_{t\geq 0}$ such that $v = H + u^{\perp}$.



Method. Use the Allen-Cahn equation with additional transport term. $\frac{\partial \varphi}{\partial t} + \tilde{u} \cdot \nabla \varphi = \Delta \varphi - \frac{W'(\varphi)}{\varepsilon^2}$

(cf. u=0: Ilmanen `93, T. `03)

Theorem (Liu-Sato-T. '10, Takasao-T. in preparation)

Suppose $n \geq 2$, $p \geq 2$, $q > 2$ satisfy

$$p > \frac{n}{2} \cdot \frac{q}{q-1}.$$

Given

- a domain $\Omega_0 \subset \mathbb{T}^n$ with C^1 boundary $M_0 = \partial\Omega_0$ and
- a vector field $u \in L^q_{loc}([0,\infty); W^{1,p}(\mathbb{T}^n))$,

there exist a family of varifolds $\{V_t\}_{t\geq 0}$ and a family of `phase boundaries' $\{\partial^*\Omega_t\}_{t\geq 0}$ which are `weak solutions' of $v = H + u^\perp$ with some good regularity property.

- For $\forall \phi \in C^1(\mathbb{T}^n;\mathbb{R}_+)$, $0 \le \forall t_1 < \forall t_2 < \infty$,
$$\int_{\mathbb{T}^n} \phi \, d\|V_t\| \Big|_{t=t_1}^{t_2} \le \int_{t_1}^{t_2} dt \int_{\mathbb{T}^n} (\nabla\phi - \phi H)\cdot \underline{(H + u^{\perp})} \, d\|V_t\|.$$
- For a.e. t, V_t is integral.
- For a.e. t, $\partial^*\Omega_t \subset \operatorname{spt}\|V_t\|$.
- There exists $T = T(M_0, \|u\|)$ such that V_t is unit density for a.e. $t < T$.
  Moreover, $\|V_t\| = \mathcal{H}^{n-1}\lfloor_{\partial^*\Omega_t}$ away from $\{\theta \ge 2\}$.
- $\exists p', \exists q'$ (expressed explicitly by p and q) with
$$\alpha := 1 - \frac{n-1}{p'} - \frac{2}{q'} > 0$$
  so that
$$\int_0^s \left(\int_{\mathbb{T}^n} |u|^{p'} \, d\|V_t\|\right)^{\frac{q'}{p'}} dt < \infty \quad , \forall s < \infty.$$

- (Local regularity results, Kasai-T. preprint `11) $\cup_{0<t<T}\partial^*\Omega_t \times \{t\} \subset \mathbb{T}^n \times (0,T)$ is locally (in space-time) $C^{1,\alpha}$ hypersurface a.e. space and a.e. time. (In fact, valid for portion away from `multiplicity ≥ 2 set ' ; cf. Allard regularity theory.)
- If u is more regular, such as C^β , then the above regularity is $C^{2,\beta}$ and $v = H + u^\perp$ is satisfied classically there (T. preprint `12).
- Note on the inequality for the exponents:

$$p = \frac{n}{2} \cdot \frac{q}{q-1} \quad \Rightarrow \quad \int_0^\infty dt \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx \right)^{\frac{q}{p}}$$ is scale invariant.

3. Ingradients of proof for the existence

For u=0, consider $\frac{\partial \varphi_\varepsilon}{\partial t} = \Delta \varphi_\varepsilon - \frac{W'(\varphi_\varepsilon)}{\varepsilon^2}$ with suitable initial data. Then for all t ≥ 0,

$$\frac{1}{\sigma}\left(\frac{\varepsilon|\nabla\varphi_\varepsilon(\cdot,t)|^2}{2} + \frac{W(\varphi_\varepsilon(\cdot,t))}{\varepsilon}\right) dx \longrightarrow \|V_t\| \qquad (\varepsilon \to 0) \qquad \sigma := \int_{-1}^{1} \sqrt{2W(s)}\, ds$$

- Huisken's monotonicity formula for MCF.
- Modica's estimate for discrepancy : $\left(\frac{\varepsilon|\nabla\varphi_\varepsilon|^2}{2} - \frac{W(\varphi_\varepsilon)}{\varepsilon}\right)_+ \leq 0$
- Analogies with Allard compactness theorem for integral varifolds to prove `integral'. (Ilmanen `93, T. `03)

For general $u \in L^q_{loc}([0,\infty); W^{1,p}(\mathbb{T}^n))$, take $u_\varepsilon \in C^\infty(\mathbb{T}^n \times [0,\infty))$ so that

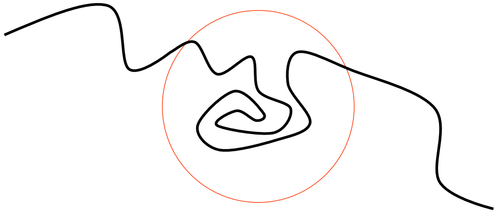
- $u_\varepsilon \to u$ in $L^q_{loc}([0,\infty); W^{1,p}(\mathbb{T}^n))$ and
- for some $0 < \gamma < 1/3, \ \sup_{\mathbb{T}^n \times [0,\infty)} |u_\varepsilon(x,t)| \le \varepsilon^{-\gamma}.$

Then consider $\dfrac{\partial \varphi_\varepsilon}{\partial t} + u_\varepsilon \cdot \nabla \varphi_\varepsilon = \Delta \varphi_\varepsilon - \dfrac{W'(\varphi_\varepsilon)}{\varepsilon^2}$ with suitable initial data.

$$\left(\frac{\varepsilon|\nabla\varphi_\varepsilon|^2}{2} - \frac{W(\varphi_\varepsilon)}{\varepsilon}\right)_+ \le C\varepsilon^{-\gamma}$$

Key estimate: $\exists C = C(M_0, \|u\|, T)$ such that

$$\sup_{(x,t)\in\mathbb{T}^n\times[0,T],\, 0<r<1} \frac{1}{r^{n-1}} \int_{B_r(x)} \left(\frac{\varepsilon|\nabla\varphi_\varepsilon(\cdot,t)|^2}{2} + \frac{W(\varphi_\varepsilon(\cdot,t))}{\varepsilon}\right) \le C.$$



4. Outlook

- Navier-Stokes-Allen-Cahn coupling (cf. Liu -Sato-T. to appear + work in progress).
- Analysis on moving singular sets.
- Existence and regularity… great, but something missing? Uniqueness.
- What is the natural notion of `dynamic stability' for MCF in varifold setting ?