Evolution and long-time behaviour of the free boundary in nonlinear Stefan problems

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International Conference on Free Boundary Problems, June 2012.

The Nonlinear Stefan Boundary Problem

$$(1) \qquad \begin{cases} u_t - d\Delta u = g(u) & \text{for } \; x \in \Omega(t), t > 0, \\ u = 0 \text{ and } u_t = \mu|\nabla_x u|^2 & \text{for } \; x \in \Gamma(t), t > 0, \\ u(0, x) = u_0(x) & \text{for } \; x \in \Omega_0, \end{cases}$$

where $\Omega(t) \subset \mathbb{R}^n$ $(n \geq 2)$ is bounded by the free boundary $\Gamma(t)$, with $\Omega(0) = \Omega_0$, μ and d are given positive constants.

For Ω_0, we assume that it is a bounded domain that agrees with the interior of its closure $\overline{\Omega}_0$, $\partial\Omega_0$ satisfies the interior ball condition, and $u_0 \in C(\overline{\Omega}_0) \cap H^1(\Omega_0)$ is positive in Ω_0 and vanishes on $\partial\Omega_0$.

For the nonlinear function g, we make the following assumptions:

$$(2) \qquad \begin{cases} \text{(i)} \ \ g(0) = 0, \\ \text{(ii)} \ \ g \in C^{1,\alpha}([0, \delta_0]) \text{ for some } \delta_0 > 0 \text{ and } \alpha \in (0,1), \\ \text{(iii)} \ \ g(u) \text{ is locally Lipschitz in } [0, \infty), \\ \text{(iv)} \ \ g(u) \leq 0 \text{ in } [M, \infty) \text{ for some } M > 0. \end{cases}$$

We note that these conditions are satisfied by standard monostable, bistable and combustion type nonlinearities.

By a result in Y. Du and Zongming Guo (JDE 2012), under these conditions (1) has a unique weak solution defined for all $t > 0$.

Main Results

Theorem 1.

1. For fixed $t > 0$, $\tilde{\Gamma}(t) := \Gamma(t) \setminus \overline{\mathrm{co}}(\Omega_0)$ is a $C^{2,\alpha}$ hypersurface in $\mathbb{R}^n$,
2. $\tilde{\Gamma} := \{(t, x) : x \in \tilde{\Gamma}(t),\ t > 0\}$ is a $C^{2,\alpha}$ hypersurface in $\mathbb{R}^{n+1}$.

In particular, the free boundary is always $C^{2,\alpha}$ smooth if Ω_0 is convex.

Here $\overline{\mathrm{co}}(\Omega_0)$ stands for the closed convex hull of Ω_0.

Theorem 2.

1. $\Omega(t)$ is expanding in the sense that $\overline{\Omega}_0 \subset \Omega(t) \subset \Omega(s)$ if $0 < t < s$.
2. $\Omega_\infty := \cup_{t>0}\Omega(t)$ is either the entire space $\mathbb{R}^n$, or it is a bounded set.
3. When $\Omega_\infty = \mathbb{R}^n$, for all large t, $\Gamma(t)$ is a smooth closed hypersurface in $\mathbb{R}^n$, and there exists a continuous function $M(t)$ such that

$$\Gamma(t) \subset \{x : M(t) - \pi d_0 \le |x| \le M(t)\}. \tag{3}$$

4. When Ω_∞ is bounded, $\lim_{t\to\infty} \|u(t,\cdot)\|_{L^\infty(\Omega(t))} = 0$.

Here d_0 is the diameter of Ω_0.

Theorem 3. If $g(u) = au - bu^2$ with a, b positive constants, then there exists $\mu^* > 0$ such that

1. $\Omega_\infty = \mathbb{R}^n$ if $\mu > \mu^*$, and Ω_∞ is bounded if $\mu \in (0, \mu^*]$;
2. when $\Omega_\infty = \mathbb{R}^n$, the following holds:

$$\lim_{t\to\infty} \frac{M(t)}{t} = k_0(\mu), \ \lim_{t\to\infty} \max_{|x|\le ct} \left| u(t,x) - \frac{a}{b} \right| = 0 \ \forall c \in (0, k_0(\mu)),$$

where $k_0(\mu)$ is a positive increasing function of μ satisfying $\lim_{\mu\to\infty} k_0(\mu) = 2\sqrt{ad}$.

Further analysis of the function $k_0(\mu)$ is given in G. Bunting, Y. Du and K. Krakowski (preprint,2011).

For example, we have

$$k_0(\mu) = \lambda_0(\frac{a\mu}{bd})\sqrt{ad}$$

with $\lambda_0(\alpha)$ satisfying

$$0 < \lambda_0(\alpha) < 2, \lim_{\alpha\to\infty} \lambda_0(\alpha) = 2, \lim_{\alpha\to 0} \frac{\lambda_0(\alpha)}{\alpha} = \frac{1}{\sqrt{3}}.$$

The following tables are obtained from numerical calculations.

α	1	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\lambda_0(\alpha)$	0.36	1.01	1.49	1.72	1.84	1.90	1.93	1.95	1.96
α	10^9	10^{10}	10^{11}	10^{12}	10^{13}	10^{14}	10^{15}	10^{16}	∞
$\lambda_0(\alpha)$	1.97	1.98	1.98	1.99	1.99	1.99	1.99	1.99	2.00

Table : $\lambda_0(\alpha)$ for $\alpha \geq 1$

α	0.01	0.1	0.2	0.3	0.4	0.5	0.7	0.9
$\lambda_0(\alpha)$	0.006	0.05	0.10	0.15	0.19	0.22	0.28	0.34
$\frac{\lambda_0(\alpha)}{\alpha}\sqrt{3}$	0.99	0.94	0.89	0.84	0.80	0.77	0.70	0.65

Table : $\lambda_0(\alpha)$ for $\alpha \leq 1$

Some Background

1. In one space dimension problem (1) with the logistic nonlinearity $g(u) = au - bu^2$ was introduced in Y. Du and Zhigui Lin (SIAM J. Math. Anal., 2010) to better understand the spreading of invasive species, where u represents the population density of the species, and the free boundary stands for the spreading front.
2. The high dimension case with radial symmetry was studied in Y. Du and Zongming Guo (JDE, 2011).

In these special cases it was proved in these papers that problem (1) exhibits a **spreading-vanishing dichotomy**:

As $t \to \infty$, either $\Omega(t)$ expands to the entire $\mathbb{R}^n$ and u converges to the positive steady-state a/b (*spreading*), or $\Omega(t)$ stays bounded and $u \to 0$ (*vanishing*).

In these cases the special geometry ensures that the free boundary and the solution are automatically smooth, which greatly simplifies the analysis.

3 In Y. Du and Zongming Guo (JDE 2012), the existence and uniqueness of a weak solution for (1) with a general Ω_0 was established.

It was shown that as $\mu \to \infty$, the weak solution of (1) converges to the solution of the corresponding Cauchy problem with initial function $\tilde{u}_0$ which is u_0 extended by 0 into the entire $\mathbb{R}^n$. Moreover, if $g(u) = au - bu^2$, it was shown in this paper that under suitable conditions on the initial values, as $t \to \infty$, $\Omega(t)$ expands to the entire space $\mathbb{R}^n$ and u converges to the positive equilibrium solution a/b, and under a set of different conditions $\Omega(t)$ remains bounded and u converges to 0. However, these two sets of conditions are not complementing to each other, and whether there is a sharp spreading-vanishing dichotomy as in the 1-d and radial cases, was left open. Our Theorem 3 here gives a complete answer to this question.

Weak Formulations of (1)

In order to obtain our results, we need two different weak formulations of (1). The first one is modeled on the approach of A. Friedman (TAMS 1968), and is given in Du-Guo (JDE 2012).

Define

$$\alpha(\xi) = \xi - d\mu^{-1}\chi_{\{\xi \le 0\}},$$

$$\tilde{u}_0 = u_0 \text{ in } \Omega_0, \; \tilde{u}_0 = 0 \text{ outside } \Omega_0.$$

For a large ball B_R and $T > 0$, if $u \in H^1((0,T) \times B_R) \cap L^\infty((0,T) \times B_R))$ satisfies in the weak sense

$$(4) \qquad \begin{cases} \partial_t[\alpha(u)] - d\Delta u = g(u) & \text{in } (0,T) \times B_R, \\ u = 0 & \text{on } (0,T) \times \partial B_R, \\ u(0,x) = \tilde{u}_0(x) & \text{in } B_R, \end{cases}$$

it is called a weak solution of (1). It can be shown that the weak solution does not depend on the choice of the large ball B_R, it is unique and agrees with the classical solution if the free boundary is smooth enough.

The idea is to approximate the discontinuous function α by a sequence of suitable smooth functions α_m, and for each approximate problem one obtains a classical solution u_m, and then show that u_m converges to a unique weak solution of (4).

For fixed $t > 0$, the set $\Omega(t) := \{x : u(t,x) > 0\}$ is contained in a compact subset of B_R. $\Gamma(t) := \partial\Omega(t)$ is the free boundary of the weak solution.

The advantage of this formulation is that comparison results follow easily from the definition and the approximation process. However, as in the classical Stefan problem, this formulation is difficult to use to obtain regularity for the free boundary.

A Second Weak Formulation

For $T > 0$ and large B_R, define $\Omega_{T,R} = (0, T) \times B_R$ and

$$f(x) = \begin{cases} u_0(x), & x \in \Omega_0, \\ -d/\mu, & x \in B_R \setminus \Omega_0. \end{cases}$$

Theorem 4. If w is a $W_2^{1,2}(\Omega_{T,R})$-solution of

$$(5) \quad \begin{cases} w_t - d\Delta w = \int_0^t g(w_t) d\tau + d\mu^{-1} \chi_{\{w \le 0\}} + f \text{ in } \Omega_{T,R}, \\ w = 0 \qquad\qquad \text{on } \partial_p(\Omega_{T,R}), \end{cases}$$

then w_t is a weak solution of (4).

Theorem 5.

1. $w \in W_p^{1,2}(\Omega_{T,R})$ for all $p > 1$,
2. $\{x : w(t,x) > 0\} = \Omega(t) := \{x : u(t,x) > 0\}$,
3. $\overline{\Omega}_0 \subset \Omega(t) \subset \Omega(s)$ if $0 < t < s$.

For the classical one-phase Stefan problem (which is roughly the case $g(u) \equiv 0$ in our setting), G. Duvaut (CRAS Paris, 1973) first introduced the weak form satisfied by $w(t,x) = \int_0^t u(\tau,x)d\tau$, and this approach was further developed by A. Friedman and D. Kinderlehrer (Indiana UMJ, 1975). This enabled the further development of the regularity theory for the one-phase Stefan problem:

- Lipschitz regularity implies $C^{1,\alpha}$ regularity [L. Caffarelli (Acta Math. 1977)].
- $C^{1,\alpha}$ regularity implies C^∞ regularity [D. Kinderlehrer and L. Nirenberg (ASNS Pisa, 1977)].

The nonlinear term $g(u)$ in (1) gives rise to a nonlocal term

$$h(t,x) = \int_0^t g(w_t(\tau,x))d\tau = \int_0^t g(u(\tau,x))d\tau$$

in (5). This causes some difficulties. In particular it is difficult to obtain a comparison principle for (5).

To overcome the difficulty with the comparison principle, we use the first weak formulation whenever comparison arguments are needed.

To adapt the regularity theory developed for the classical one-phase Stefan problem to (5), we use the following result.

Lemma 1. Near a free boundary point $(t_0, x_0) \in \partial\{w > 0\}$, $h(t, x)$ is close to 0 and moreover,

1. h is Hölder continuous in $\{w > 0\}$ near (t_0, x_0) if the free boundary is Lipschitz continuous,
2. h is Lipschitz near (t_0, x_0) if the free boundary is C^1,
3. h is $C^{1,\alpha}$ near (t_0, x_0) if further g is $C^{1,\alpha}$ near 0 and $u = w_t$ is $C^{1,\alpha}$ near (t_0, x_0).

Regularity Results

Using Lemma 1 and with a lot of effort, we can prove

Proposition 1. If g satisfies (i), (iii) and (iv) in (2), and if the free boundary is Lipschitz near (t_0, x_0), then the free boundary is $C^{1,\gamma}$ near (t_0, x_0) for any $\gamma \in (0,1)$. If g also satisfies (ii) in (2), then the free boundary is $C^{2,\alpha}$ near (t_0, x_0).

It remains to prove the Lipschitz regularity of the free boundary outside $\overline{\mathrm{co}}(\Omega_0)$. To do this, we use a monotonicity method, which relies on a reflection and comparison technique, similar to the moving plane method in elliptic problems. Such a technique was first used in parabolic problems by Aronson-Caffarelli (TAMS 1983), C.K.R.T. Jones (Rocky Mountain JM, 1983) and H. Matano (Conf. Proceedings, 1983).

For any given $z_0 \notin \overline{\mathrm{co}}(\Omega_0)$, we can associate a uniquely determined open set of unit vectors S_{z_0} and an open cone C_{z_0} with vertex 0 in the following way:

$$S_{z_0} := \{\nu \in \mathbb{R}^N : \ |\nu| = 1 \ , \ \nu \cdot (x - z_0) < 0 \ \forall x \in \overline{\mathrm{co}}(\Omega_0)\},$$

$$C_{z_0} := \{\lambda\nu : \lambda \in (0,1), \ \nu \in S_{z_0}\}.$$

C_{z_0} has the following geometric characterization: For any $x \in z_0 + C_{z_0}$, the straight line l_0 passing through z_0 and x must intersect $\mathrm{co}(\Omega_0)$, and the plane passing through z_0 normal to l_0 does not intersect $\overline{\mathrm{co}}(\Omega_0)$.

Lemma 2. For $(s, z) \in (0, T) \times [B_R \setminus \overline{\mathrm{co}}(\Omega_0)]$, and all $\nu \in S_z$, we have $\partial_\nu u(s, z) \le 0$. Moreover, for every $s_0 \in (0, T)$, $z_0 \in \Omega(s_0) \setminus \overline{\mathrm{co}}(\Omega_0)$ and $\nu \in S_{z_0}$, we have $\partial_\nu u(s_0, z_0) < 0$.

Making use of Lemma 2, we can show

Lemma 3. Suppose that $t_0 \in (0, T)$, $x_0 \in \Gamma(t_0) \setminus \overline{\mathrm{co}}(\Omega_0)$ and $\delta > 0$ is small. Then there exists $\epsilon > 0$ small such that $u(t_0, x) \equiv 0$ in $(x_0 + C^\delta_{x_0}) \cap B_\epsilon(x_0)$, and $u(t_0, x) > 0$ in $(x_0 - C^\delta_{x_0}) \cap B_\epsilon(x_0)$. Here $C^\delta_{x_0} \subset C_{x_0}$ is a cone with vertex 0 and the same axis as C_{x_0}, and $B_\epsilon(x_0) = \{|x - x_0| < \epsilon\}$.

By Lemma 3, it is easy to show that $\Gamma(t_0)$ is the graph of a Lipschitz function near x_0.

Using (5) it can further be proved that near (t_0, x_0), $\partial\{w > 0\}$ is the graph of a Lipschitz function with parabolic distance

$$|(t,x) - (s,y)| = \sqrt{|t-s| + |x-y|^2}.$$

Thus we have proved the following result.

Proposition 2. If $(t_0, x_0) \in \partial\{w > 0\}$ with $t_0 > 0$ and $x_0 \notin \overline{\text{co}}(\Omega_0)$, then $\partial\{w > 0\}$ is Lipschitz near (t_0, x_0).

Clearly Theorem 1 is a consequence of Propositions 1 and 2.

Global Properties and Dichotomy

Using Lemma 2, it is not difficult to prove the following result.

Theorem 6. At any point $x_0 \in \Gamma(t) \setminus \overline{\mathrm{co}}(\Omega_0)$, the inward normal line to $\Gamma(t)$ at x_0 intersects $\overline{\mathrm{co}}(\Omega_0)$. Moreover, if ℓ is any ray emanating from Ω_0, then $\ell \cap \big(\Gamma(t) \setminus \overline{\mathrm{co}}(\Omega_0)\big)$ contains at most one point.

One important consequence of this result is

Theorem 7. Let x_* be any point in Ω_0 and put

$$m(t) = \min_{x \in \Gamma(t) \setminus \overline{\mathrm{co}}(\Omega_0)} |x - x_*|, \quad M(t) = \max_{x \in \Gamma(t)} |x - x_*|.$$

Suppose $B_{R_0}(x_*) \supset \overline{\mathrm{co}}(\Omega_0)$, and there exists $t_0 > 0$ such that $M(t_0) > (2\pi + 1)R_0$. Then $m(t) > M(t) - 2\pi R_0$ for all $t \geq t_0$. Hence for $t \geq t_0$, $\tilde{\Gamma}(t) := \Gamma(t) \setminus \overline{\mathrm{co}}(\Omega_0)$ is a $C^{2,\alpha}$ closed hypersurface in $\mathbb{R}^n$ satisfying

$$\tilde{\Gamma}(t) \subset \{x \in \mathbb{R}^n : M(t) - 2\pi R_0 < |x - x_*| \leq M(t)\}.$$

If Ω_0 is convex, then $\Gamma(t)$ is always $C^{2,\alpha}$ (for $t > 0$), and Theorem 7 implies that

$$\Omega_\infty = \cup_{t>0}\Omega(t)$$

is either the entire R^n, or it is a bounded set.

If Ω_0 is not convex, we have the following result.

Theorem 8. In the case that Ω_∞ is unbounded and Ω_0 is not convex, there is a $T_0 > 0$, such that for all $t \geq T_0$,

$$\overline{\mathrm{co}}(\Omega_0) \subset \Omega(t).$$

Thus Ω_∞ unbounded implies $\Omega_\infty = \mathbb{R}^n$.

If Ω_∞ is bounded, we can choose a big ball B_R such that $\overline{\Omega}_\infty \subset B_R$. We then define

$$E(t) = \int_{B_R} [\frac{d}{2}|\nabla u|^2 - G(u)]dx, \;\; G(u) = \int_0^u g(\xi)d\xi.$$

We have the following energy inequality

$$E(t_2) - E(t_1) \le -\int_{t_1}^{t_2} \int_{B_R} |u_t|^2 dxdt$$

for the weak solution u of (1) and $0 < t_1 < t_2 < \infty$.

Making use of this energy inequality we can prove

Theorem 9. If Ω_∞ is bounded, then $u(t, \cdot) \to 0$ uniformly as $t \to \infty$.

Theorem 2 clearly follows from Theorems 5, 7, 8 and 9.

Finally Theorem 3 is proved by making use of suitable comparison principles (by the first weak formulation), results on the radial case in Du-Guo (JDE 2011), and the results obtained above. To obtain the threshold number μ^*, we also need the continuous dependence of the weak solution on μ, for which we use the second weak formulation.

Related Recent Research

(a) Free boundary problem with logistic nonlinearity:

- time-periodic environment with spatial radial symmetry [Y. Du, Zongming Guo and Rui Peng, preprint, 2011],
- seasonal succession in one space dimension [R. Peng and X.Q. Zhao, preprint, 2011],
- spatially-periodic environment [Maolin Zhou (2012), Y. Du and Xing Liang (2012)].

(b) Other nonlinearities and systems:

- Homogeneous one space dimension case with general monostable, bistable and combustion type nonlinearities [Y. Du and Bendong Lou, preprint, 2011].
- Homogeneous one space dimension case with Dirichlet and free boundary conditions for monostable and bistable nonlinearities [Y. Kaneko and Y. Yamade, preprint, 2011].
- Homogeneous one space dimension free boundary system for two competing species with various free boundary conditions [C-H Chang and C-C Chen (2011), J.S. Guo and C.H. Wu (2012), Y. Du and Zhigui Lin (work in progress)].
- Monostable and bistable nonlinearity with modified free boundary conditions [Bendong Lou and students, work in progress].
- Free boundary problems with various super-linear nonlinearities, or in systems of two species without consideration of spreading speed [Z.G. Lin and collaborators, a series of papers from 2007].

Thank You!