An optimization based numerical approach for free boundary problems modeled by variational inequalities of the second kind

Juan Carlos De Los Reyes

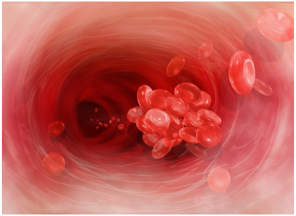
Research Group on Numerical Optimization and Scientific Computing
Departamento de Matemática
Escuela Politécnica Nacional de Quito, Ecuador

Free Boundary Problems, June 2012

What is a viscoplastic (Bingham) flow?

What is a viscoplastic (Bingham) flow?

What is a viscoplastic (Bingham) flow?



Mathematical model

Boundary value problem

$$
\begin{aligned}
-\operatorname{Div}\sigma + (y\cdot\nabla)y + \nabla\phi &= f && \text{in } \Omega\\
\operatorname{div} y &= 0, && \text{in } \Omega\\
y &= 0, && \text{on } \Gamma,\\
\sigma &= 2\mu\mathcal{E}y
\end{aligned}
$$

y : velocity vector field
$\mathcal{E}$: rate of strain tensor

ϕ : pressure
f : volume force
μ : viscosity coefficient

Mathematical model

Boundary value problem

$$
\begin{aligned}
&-\operatorname{Div}\sigma + (y\cdot\nabla)y + \nabla\phi = f && \text{in } \Omega\\
&\operatorname{div} y = 0, && \text{in } \Omega\\
&y = 0, && \text{on } \Gamma,\\
&\sigma = 2\mu\mathcal{E}y + g\frac{\mathcal{E}y}{|\mathcal{E}y|}, && \text{if } \mathcal{E}y \neq 0,\\
&|\sigma| \leq g, && \text{if } \mathcal{E}y = 0,
\end{aligned}
$$

y : velocity vector field

$\mathcal{E}$: rate of strain tensor

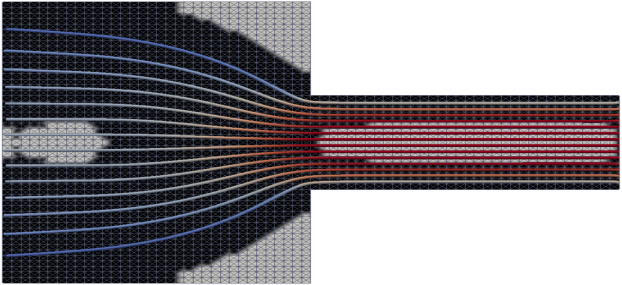
g : plasticity threshold

ϕ : pressure

f : volume force

μ : viscosity coefficient

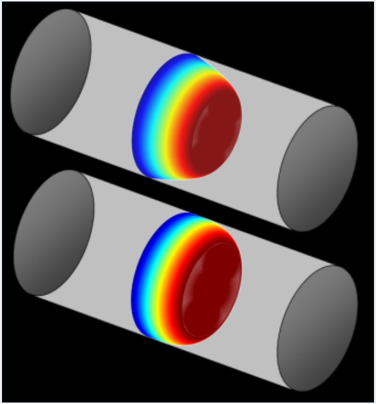
Challenges in the numerical simulation

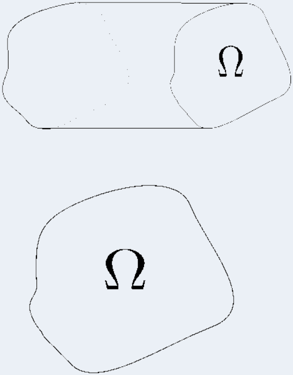


Identification of fluid zones, rigid solid motion zones and stagnation zones.

Simplified case

Pipe of cross section Ω





Simplified mathematical model

Energy minimization (Mosolov-Miasnikov (1965))

$$\min_{y(x)\in H_0^1(\Omega)} \int_\Omega |\nabla y|^2\, dx + g \int_\Omega |\nabla y|\, dx - \int_\Omega f \cdot y\, dx$$

Simplified mathematical model

Energy minimization (Mosolov-Miasnikov (1965))

$$\min_{y(x)\in H_0^1(\Omega)} \int_\Omega |\nabla y|^2\, dx + g \int_\Omega |\nabla y|\, dx - \int_\Omega f \cdot y\, dx$$

Convex nondifferentiable term!

Simplified mathematical model

Energy minimization (Mosolov-Miasnikov (1965))

$$\min_{y(x)\in H_0^1(\Omega)} \int_\Omega |\nabla y|^2\, dx + g\int_\Omega |\nabla y|\, dx - \int_\Omega f\cdot y\, dx$$

Convex nondifferentiable term!

Variational inequality (necessary and sufficient condition)

$$a(y, v-y)+g\int_\Omega |\nabla v|dx-g\int_\Omega |\nabla y|dx \geq \int_\Omega f(v-y)dx, \forall v \in H_0^1(\Omega)$$

where $a(y, w) := \int_\Omega \nabla y^T \nabla w\, dx$.

Duality

Primal problem

$$\inf_{y \in H_0^1(\Omega)} J(y) = \frac{1}{2} a(y, y) + g \int_\Omega |\nabla y| \, dx - \int_\Omega f \cdot y \, dx.$$

Duality

Primal problem

$$\inf_{y \in H_0^1(\Omega)} J(y) = \frac{1}{2} a(y,y) + g \int_\Omega |\nabla y| \, dx - \int_\Omega f \cdot y \, dx.$$

$$\Updownarrow$$

Dual Problem

$$\sup_{|q(x)| \leq g} \; -\frac{1}{2} a(y,y)$$

subject to:

$$a(y,v) + (q, \nabla v) = (f,v), \text{ for all } v \in H_0^1(\Omega)$$

Some references

Primal approach: direct global regularization

Glowinski-Lions-Tremolieres (1976), Glowinski (1984), Frigaard-Nouar (2005), Dean-Glowinski-Guidoboni (2007),...

Some references

Primal approach: direct global regularization

Glowinski-Lions-Tremolieres (1976), Glowinski (1984), Frigaard-Nouar (2005), Dean-Glowinski-Guidoboni (2007),...

Drawbacks: physical properties not reflected, poor identification of free boundary, ill-conditioned for large parameter values.

Some references

Primal approach: direct global regularization

Glowinski-Lions-Tremolieres (1976), Glowinski (1984), Frigaard-Nouar (2005), Dean-Glowinski-Guidoboni (2007),...

Drawbacks: physical properties not reflected, poor identification of free boundary, ill-conditioned for large parameter values.

Multiplier approach: use of dual information

Glowinski (1984), Glowinski-Le Tallec (1989), Sánchez (1998), Roquet-Saramito (2003,2008), Huilgol-You (2005), Dean et al. (2007), Muravleva-Muravleva (2009), Olshanskii (2009)

Some references

Primal approach: direct global regularization

Glowinski-Lions-Tremolieres (1976), Glowinski (1984), Frigaard-Nouar (2005), Dean-Glowinski-Guidoboni (2007),...

Drawbacks: physical properties not reflected, poor identification of free boundary, ill-conditioned for large parameter values.

Multiplier approach: use of dual information

Glowinski (1984), Glowinski-Le Tallec (1989), Sánchez (1998), Roquet-Saramito (2003,2008), Huilgol-You (2005), Dean et al. (2007), Muravleva-Muravleva (2009), Olshanskii (2009)

Drawback: use of rather slow methods.

Some references

Primal approach: direct global regularization

Glowinski-Lions-Tremolieres (1976), Glowinski (1984), Frigaard-Nouar (2005), Dean-Glowinski-Guidoboni (2007),...

Drawbacks: physical properties not reflected, poor identification of free boundary, ill-conditioned for large parameter values.

Multiplier approach: use of dual information

Glowinski (1984), Glowinski-Le Tallec (1989), Sánchez (1998), Roquet-Saramito (2003,2008), Huilgol-You (2005), Dean et al. (2007), Muravleva-Muravleva (2009), Olshanskii (2009)

Drawback: use of rather slow methods.

Guiding Idea: design Newton type algorithms in combination with multiplier approach

Tikhonov's Regularization

Dual Problem

$$\begin{cases} \displaystyle\min_{|q(x)|\leq g} \frac{1}{2} a(y,y) \\ \text{subject to:} \\ a(y,v) + (q, \nabla v) = (f,v), \ \text{ for all } v \in H_0^1(\Omega) \end{cases}$$

Tikhonov's Regularization

Dual Problem

$$\begin{cases} \min\limits_{|q(x)|\leq g} \frac{1}{2}a(y,y) \\ \text{subject to:} \\ a(y,v) + (q,\nabla v) = (f,v), \ \text{for all } v \in H_0^1(\Omega) \end{cases}$$

No unique solution!

Tikhonov's Regularization

Penalized Dual Problem

$$\begin{cases} \min\limits_{|q(x)|\leq g} \frac{1}{2}a(y,y) + \frac{1}{2\gamma}\|q\|^2_{\mathbb{L}^2} \\ \text{subject to:} \\ a(y,v) + (q, \nabla v) = (f,v), \ \text{for all } v \in H^1_0(\Omega) \end{cases}$$

Tikhonov's Regularization

Penalized Dual Problem

$$\begin{cases} \min_{|q(x)| \le g} \frac{1}{2} a(y,y) + \frac{1}{2\gamma} \|q\|^2_{\mathbb{L}^2} \\ \text{subject to:} \\ a(y,v) + (q, \nabla v) = (f, v), \ \text{for all } v \in H^1_0(\Omega) \end{cases}$$

Theorem

There exists a unique solution $(q_\gamma, y_\gamma) \in \mathbb{L}^2(\Omega) \times H^1_0(\Omega)$ to the penalized dual problem.

Theorem

The regularized dual solutions q_γ *converge to a solution* $\overline{q}$ *weakly in* $\mathbb{L}^2(\Omega)$ *as* $\gamma \to \infty$*. Moreover, the correspondent primal solutions* y_γ *converge to the original solution* $\overline{y}$ *strongly in* $H_0^1(\Omega)$ *as* $\gamma \to \infty$*.*

Theorem

The regularized dual solutions q_γ converge to a solution $\overline{q}$ weakly in $\mathbb{L}^2(\Omega)$ as $\gamma \to \infty$. Moreover, the correspondent primal solutions y_γ converge to the original solution $\overline{y}$ strongly in $H_0^1(\Omega)$ as $\gamma \to \infty$.

Regularized optimality system

$$a(y_\gamma, v) + (q_\gamma, \nabla v) = (f, v), \text{ for all } v \in H_0^1(\Omega)$$
$$\max\left(g, \gamma|\nabla(y_\gamma)|\right) q_\gamma = g\gamma\nabla(y_\gamma), \quad \text{for } \gamma > 0.$$

Theorem

The regularized dual solutions q_γ *converge to a solution* $\overline{q}$ *weakly in* $\mathbb{L}^2(\Omega)$ *as* $\gamma \to \infty$. *Moreover, the correspondent primal solutions* y_γ *converge to the original solution* $\overline{y}$ *strongly in* $H_0^1(\Omega)$ *as* $\gamma \to \infty$.

Regularized optimality system

$$a(y_\gamma, v) + (q_\gamma, \nabla v) = (f, v), \text{ for all } v \in H_0^1(\Omega)$$
$$\max\,(g, \gamma|\nabla(y_\gamma)|)\, q_\gamma = g\gamma\nabla(y_\gamma), \quad \text{for } \gamma > 0.$$

Difficulty for Newton type algorithm: *max* function is not differentiable!

Semismooth Newton method

Definition (Newton differentiability)

If there exists a neighborhood $N(x^*) \subset S$ and a family of mappings $G : N(x^*) \to \mathcal{L}(X, Y)$ such that

$$\lim_{\|h\|_X \to 0} \frac{\|\mathcal{F}(x^* + h) - \mathcal{F}(x^*) - G(x^* + h)(h)\|_Y}{\|h\|_X} = 0,$$

then $\mathcal{F}$ is called Newton differentiable at x^*.

Semismooth Newton method

Definition (Newton differentiability)

If there exists a neighborhood $N(x^*) \subset S$ and a family of mappings $G : N(x^*) \to \mathcal{L}(X, Y)$ such that

$$\lim_{\|h\|_X \to 0} \frac{\|\mathcal{F}(x^* + h) - \mathcal{F}(x^*) - G(x^* + h)(h)\|_Y}{\|h\|_X} = 0,$$

then $\mathcal{F}$ is called Newton differentiable at x^*.

Semi-smooth Newton step

$$x^{k+1} = x^k - G(x^k)^{-1}\mathcal{F}(x^k).$$

References: Hintermüller-Ito-Kunisch (2003), M. Ulbrich (2003).

Differentiability of the *max* function

The mapping $y \mapsto max(0, y)$ from $\mathbb{R}^n \to \mathbb{R}^n$ with

$$g(y) = \begin{cases} 1 \text{ if } y \geq 0 \\ 0 \text{ if } y < 0 \end{cases}$$

as generalized derivative, is Newton differentiable.

Differentiability of the *max* function

The mapping $y \mapsto max(0, y)$ from $\mathbb{R}^n \to \mathbb{R}^n$ with

$$g(y) = \begin{cases} 1 \text{ if } y \geq 0 \\ 0 \text{ if } y < 0 \end{cases}$$

as generalized derivative, is Newton differentiable.

In function space

The mapping $max(0, \cdot)$ from $L^q(\Omega) \to L^p(\Omega)$, with $1 \leq p < q \leq \infty$ and

$$g(v)(x) = \begin{cases} 1 \text{ if } v(x) \geq 0 \\ 0 \text{ if } v(x) < 0 \end{cases}$$

as generalized derivative, is Newton differentiable.

Algorithm for discretized problem

No regularity gain $\Rightarrow$ finite dimensional analysis

Algorithm for discretized problem

No regularity gain $\Rightarrow$ finite dimensional analysis

Convergence

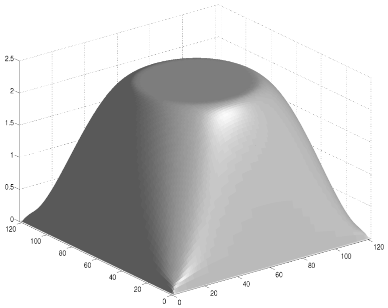
If (y_h^0, q_h^0) is sufficiently close to (y_h, q_h) then the iterates (y_h^k, q_h^k) converge superlinearly to (y_h, q_h) .

Globalization based on modified Jacobian:

M. Hintermüller and G. Stadler.
An infeasible primal-dual algorithm for TV-based inf-convolution-type image restoration. *SIAM Journal on Scientific Computing*, 28 (1), pp. 1-23, 2006.

Numerics

- Data: $\Omega = (0,1)^2$, $g = 1$, $\mu = 0.1$, $\gamma = 10^3$ and $f = 10$.
- Finite differences, centered differences for the gradient



- Superlinear convergence
- Very accurate determination of solid-fluid zones.

Extension to 2d Bingham flow model

Stationary model

$$\min_{y \in V} \mu \int_\Omega |\mathcal{E}(y)|^2 \, dx + g \int_\Omega |\mathcal{E}(v)| \, dx - \int_\Omega f \cdot y \, dx$$

where $V := \{v \in \mathbb{H}_0^1 : \mathrm{div}\, v = 0\}$, $\mathcal{E}(v) = \frac{1}{2}\left(\nabla v + (\nabla v)^T\right)$

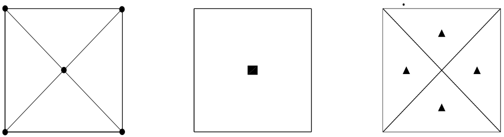
Extension to 2d Bingham flow model

Stationary model

$$\min_{y \in V} \ \mu \int_\Omega |\mathcal{E}(y)|^2 \, dx + g \int_\Omega |\mathcal{E}(v)| \, dx - \int_\Omega f \cdot y \, dx$$

where $V := \{v \in \mathbb{H}_0^1 : \mathrm{div}\ v = 0\}$, $\mathcal{E}(v) = \frac{1}{2}\left(\nabla v + (\nabla v)^T\right)$

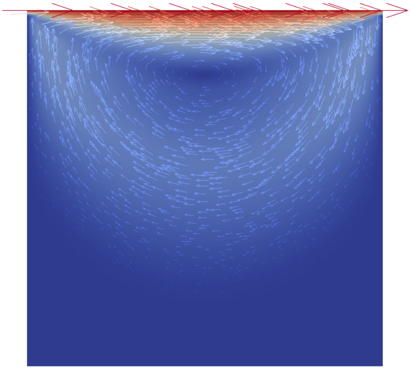
Discretization (cross-grid $\mathbb{P}_1$)-$\mathbb{Q}_0$ elements

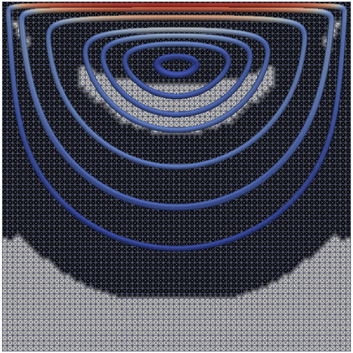


De Los R. and S. González.
Numerical simulation of two-dimensional Bingham fluid flow by semismooth Newton methods. *Journal of Computational and Applied Mathematics*, 2010.

Driven cavity flow

Data: $\Omega =]0,1[^2$, $g = 2.5$, $\mu = 1$, $\gamma = 10^3$ and $f = 0$.





Time-dependent convective problem

Regularized system

$$\partial_t \mathbf{y}_\gamma(t) - \operatorname{Div} \Delta \mathbf{y}_\gamma(t) - \operatorname{Div} \mathbf{q}_\gamma(t) + (\mathbf{y}(t) \cdot \nabla)\mathbf{y}(t) + \nabla p(t) = \mathbf{f}(t)$$

$$\operatorname{div} \mathbf{y}_\gamma(t) = 0,$$

$$\max\left(\frac{g}{\gamma}, \|\mathcal{E}\mathbf{y}_\gamma(x,t)\|\right) \mathbf{q}_\gamma(x,t) = g\mathcal{E}\mathbf{y}_\gamma(x,t), \text{a.e. in } Q, \gamma > 0,$$

+ I.C. and B.C..

Property: $\|\mathbf{q}_\gamma(x,t)\| \leq g$ a.e. in Q.

Suitability of the Regularized-Multiplier Approach

Theorem

There exists a unique solution $\mathbf{y}_\gamma \in L^2(0, T; V)$ *for the proposed regularized system of equations, for an appropriate initial condition* $\mathbf{y}_0$*.*

Suitability of the Regularized-Multiplier Approach

Theorem

There exists a unique solution $\mathbf{y}_\gamma \in L^2(0,T;V)$ *for the proposed regularized system of equations, for an appropriate initial condition* $\mathbf{y}_0$.

Theorem

The regularized solutions $(\mathbf{y}_\gamma, \mathbf{q}_\gamma)$ *converge to the original solution* $(\mathbf{y}, \overline{\mathbf{q}})$*, as* $\gamma \to \infty$*, in the sense that*

$$\int_Q |\mathbf{y}_\gamma - \mathbf{y}|^2 + \int_Q \|\nabla(\mathbf{y}_\gamma - \mathbf{y})\|^2 \to 0, \ \textit{and}$$

$$\mathbf{q}_\gamma \rightharpoonup \overline{\mathbf{q}} \ \textit{weakly in} \ L^2(\mathbb{L}^{2\times 2}).$$

Semi-Discretized Regularized-Multiplier System

$$\mathbf{M}^h \frac{\partial}{\partial t}\vec{\mathbf{y}}(t) + \mathbf{A}_\mu^h \vec{\mathbf{y}}(t) + \mathbf{Q}^h \vec{\mathbf{q}}(t) + \mathbf{C}^h(\vec{\mathbf{y}}(t))\,\vec{\mathbf{y}}(t) + B^h \vec{p}(t) = \vec{\mathbf{f}}(t)$$

$$-(B^h)^\top \vec{\mathbf{y}}(t) = 0$$

$$\max\left(\frac{g}{\gamma}, N(\mathcal{E}^h \mathbf{y}(\vec{t}))\right) \star \mathbf{q}(\vec{t}) = g\mathcal{E}^h \mathbf{y}(\vec{t}),$$

+ I.C.

where $\mathbf{C}^h(\vec{\mathbf{w}})$ is the F.E.M. matrix associated with the nonlinear form $(\mathbf{y}(t) \cdot \nabla)\mathbf{y}(t)$.

Time-Discretization

Backward differentiation formulae

When applied to $y' = \Psi(y)$, the BDF2 scheme reads as:

$$y^{k+2} - \frac{4}{3}\, y^{k+1} + \frac{1}{3}\, y^{k} = \frac{2}{3}\, k\, \Psi^{k+2}, \quad \text{for } k \leq \mathcal{N} - 2$$

where y^k: approximation of y at each time step k.

Time-Discretization

Backward differentiation formulae

When applied to $y' = \Psi(y)$, the BDF2 scheme reads as:

$$y^{k+2} - \frac{4}{3}\, y^{k+1} + \frac{1}{3}\, y^{k} = \frac{2}{3}\, k\, \Psi^{k+2}, \quad \text{for } k \leq \mathcal{N} - 2$$

where y^k: approximation of y at each time step k.

Property: Second order in time

Time-Discretization

Backward differentiation formulae

When applied to $y' = \Psi(y)$, the BDF2 scheme reads as:

$$y^{k+2} - \frac{4}{3}\, y^{k+1} + \frac{1}{3}\, y^{k} = \frac{2}{3}\, k\, \Psi^{k+2}, \quad \text{for } k \leq \mathcal{N} - 2$$

where y^k: approximation of y at each time step k.

Property: Second order in time

BDF2: multistep method $\Rightarrow$ requires initialization for y^0 and y^1.

Fully-Discretized Regularized-Multiplier System

BDF2 discretization: at $t_{k+1} = (k+1)\delta t$, for $k = 0, \dots, \mathcal{N} - 1$:

$$\left(\frac{3}{2\delta t}\mathbf{M}^h + \mathbf{A}^h_\mu\right)\, \vec{\mathbf{y}}^{\,k+2} + \mathbf{Q}^h \vec{\mathbf{q}}^{\,k+2} + B^h \vec{p}^{\,k+2} = \widetilde{\mathbf{F}}^{\,k+2}$$

$$-(B^h)^\top \vec{\mathbf{y}}^{\,k+2} = 0$$

$$\max\left(\frac{g}{\gamma}, N(\mathcal{E}^h \vec{\mathbf{y}}^{\,k+2})\right) \star \mathbf{q}^{\vec{k+2}} = g\mathcal{E}^h \vec{\mathbf{y}}^{\,k+2}.$$

$\widetilde{\mathbf{F}}^{\,k+2} := \vec{\mathbf{f}}^{\,k+2} - \mathbf{C}^h(\vec{\tilde{\mathbf{y}}}^{\,k})\vec{\tilde{\mathbf{y}}}^{\,k} + \mathbf{M}^h\left(\frac{2}{\delta t}\vec{\mathbf{y}}^{\,k+1} - \frac{1}{2\delta t}\vec{\mathbf{y}}^{\,k}\right)$ and
$\vec{\tilde{\mathbf{y}}} := 2\vec{\mathbf{y}}^{\,k+1} - \vec{\mathbf{y}}^{\,k}$

Here $\vec{\mathbf{y}}^{\,0} \Rightarrow$ I.C. and $\vec{\mathbf{y}}^{\,1} \Rightarrow$ implicit Euler.

G.A. Baker, V.A. Dougalis and O.A. Karakashian
On a Higher Order Accuracy Fully Discrete Galerkin Approximation to the Navier-Stokes Equations.
Mathematics of Computation., 1982.

Flow driven cavity.

$\Omega := (0,1)^2$, plasticity threshold $g = 2.5$ and viscosity $\mu = 1$.

Flow driven cavity.

$\Omega := (0,1)^2$, plasticity threshold $g = 2.5$ and viscosity $\mu = 1$.
Mesh information: $h = \frac{1}{300}$ and $\delta t = 0.001 (\approx 0.2 * (h^{4/5}))$

Figure: Vector velocity field (left) and Rigid-Plastic zones (right)

What about mesh independence?

Regularization

Regularized dual

$$\begin{cases} \min\limits_{|q(x)|\leq g} \frac{1}{2} a(y,y) \\ \text{subject to:} \\ a(y,v) + (q, \nabla v) = (f,v), \ \text{for all } v \in H_0^1(\Omega) \end{cases}$$

Regularization

Regularized dual

$$\begin{cases} \min & \frac{1}{2}a(y,y) + \frac{\gamma}{2}\|(|q|-g)^+\|^2_{L^2(\Omega)} \\ \text{subject to:} & \\ a(y,v)+(q,\nabla v)=(f,v), & \text{for all } v \in H^1_0(\Omega) \end{cases}$$

where $\gamma > 0$, $(\cdot)^+ = \max(0,\cdot)$.

Regularization

Regularized dual

$$\begin{cases} \min \quad \frac{1}{2}a(y,y) + \frac{\gamma}{2}\|(|q|-g)^+\|^2_{L^2(\Omega)} + \frac{1}{2\gamma}\|q\|^2_{\mathbb{H}^1_0(\Omega)} \\ \text{subject to:} \\ a(y,v) + (q,\nabla v) = (f,v), \text{ for all } v \in H^1_0(\Omega) \end{cases}$$

where $\gamma > 0$, $(\cdot)^+ = \max(0,\cdot)$.

Theorem (Convergence as $\gamma \to \infty$)

The solutions $\{q_\gamma\}$ to the regularized dual problem converge to the original solution q weakly in $\mathbb{L}^2(\Omega)$ as $\gamma \to \infty$ and

$$\text{div}\, q_\gamma \to \text{div}\, q \text{ strongly in } \mathbb{H}^{-1}(\Omega) \text{ as } \gamma \to \infty.$$

Moreover, the correspondent primal solutions y_γ converge to the original solution $\bar{y}$ strongly in $H_0^1(\Omega)$ as $\gamma \to \infty$.

Theorem (Convergence as $\gamma \to \infty$)

The solutions $\{q_\gamma\}$ to the regularized dual problem converge to the original solution q weakly in $\mathbb{L}^2(\Omega)$ as $\gamma \to \infty$ and

$$div\ q_\gamma \to div\ q \text{ strongly in } \mathbb{H}^{-1}(\Omega) \text{ as } \gamma \to \infty.$$

Moreover, the correspondent primal solutions y_γ converge to the original solution $\overline{y}$ strongly in $H_0^1(\Omega)$ as $\gamma \to \infty$.

Optimality system

$$-\mu \Delta y_\gamma - \text{div } q_\gamma = f$$

$$\nabla y_\gamma - \frac{1}{\gamma} \vec{\Delta} q_\gamma + \max(0, \gamma(|q_\gamma| - g)) \frac{q_\gamma}{|q_\gamma|} = 0$$

Nonsmooth system

Reformulation as operator equation

$$W(q_\gamma, y_\gamma) = \begin{pmatrix} -\mu \Delta y_\gamma - \text{div } q_\gamma - f \\ \nabla y_\gamma - \frac{1}{\gamma} \vec{\Delta} q_\gamma + \gamma \max(0, |q_\gamma| - g) \frac{q_\gamma}{|q_\gamma|} \end{pmatrix} = 0.$$

Nonsmooth system

Reformulation as operator equation

$$W(q_\gamma, y_\gamma) = \begin{pmatrix} -\mu \Delta y_\gamma - \operatorname{div} q_\gamma - f \\ \nabla y_\gamma - \frac{1}{\gamma} \overrightarrow{\Delta} q_\gamma + \gamma \max(0, |q_\gamma| - g) \frac{q_\gamma}{|q_\gamma|} \end{pmatrix} = 0.$$

max function is not differentiable
$\Longrightarrow$ semismooth Newton method

Semismoothness

The mapping

$$q \mapsto (|q| - g)^+ \frac{q}{|q|}$$

is Newton differentiable from $\mathbb{L}^q(\Omega) \to \mathbb{L}^p(\Omega),\ q > p$ with derivative

$$\mathcal{M}(q) = \chi_A(q) \frac{qq^T}{|q|} + (|q| - g)^+ \frac{1}{|q|}(id + \frac{qq^T}{|q|^2}).$$

Here $\chi_A(q)$ denotes the characteristic function of

$$\mathcal{A}(q) = \{x \in S : |q(x)| > \bar{g}\}.$$

SSN Algorithm

(i) Choose a $q^0 \in H$; set $k := 0$

(ii) Solve

$$\begin{pmatrix} -\mu\Delta & -\text{div} \\ \nabla & -\frac{1}{\gamma}\overrightarrow{\Delta} + \gamma\mathcal{M}(q_k) \end{pmatrix} \begin{pmatrix} \delta_y \\ \delta_q \end{pmatrix} = \begin{pmatrix} \mu\Delta y_k + \text{div}\; q_k + f \\ -\nabla y_k + \frac{1}{\gamma}\overrightarrow{\Delta} q_k + \gamma \max(0, |q_k| - g)\, \frac{q_k}{|q_k|} \end{pmatrix}$$

(iii) Set $q^{k+1} = q^k + \delta_q, y^{k+1} = y^k + \delta_y^k$ and $k = k + 1$. Return to (ii).

Convergence

Theorem (Local superlinear convergence)

The Newton derivative operator is uniformly invertible. If q^0 is sufficiently close to q_γ, then the generalized Newton iteration is well-defined and satisfies

$$\|q^{k+1} - q_\gamma\|_{\mathbb{L}^2(\Omega)} = o(\|q^k - q_\gamma\|_{\mathbb{L}^2(\Omega)}) \text{ as } k \to \infty.$$

Convergence

Theorem (Local superlinear convergence)

The Newton derivative operator is uniformly invertible. If q^0 is sufficiently close to q_γ, then the generalized Newton iteration is well-defined and satisfies

$$\|q^{k+1} - q_\gamma\|_{\mathbb{L}^2(\Omega)} = o(\|q^k - q_\gamma\|_{\mathbb{L}^2(\Omega)}) \text{ as } k \to \infty.$$

Theorem (Global convergence)

The Newton direction is a descent direction and (with a suitable line search rule) the method converges globally.

Convergence behavior

Superlinear convergence

Iteration	$\mid \mathcal{A}_k \mid$	increment	rate
1	12800	0.3551	-
2	6518	1.3968	3.934147
3	11456	0.4949	0.354309
$\vdots$	$\vdots$	$\vdots$	$\vdots$
11	9932	0.001176	0.088149
12	9928	2.2032e-5	0.018734
13	9928	1.8242e-9	0.000083

Convergence behavior

Superlinear convergence

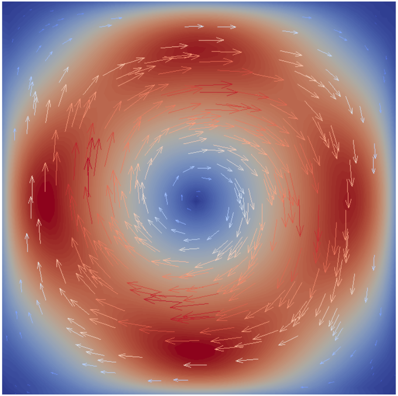
Iteration	$\mid \mathcal{A}_k \mid$	increment	rate
1	12800	0.3551	-
2	6518	1.3968	3.934147
3	11456	0.4949	0.354309
⋮	⋮	⋮	⋮
11	9932	0.001176	0.088149
12	9928	2.2032e-5	0.018734
13	9928	1.8242e-9	0.000083

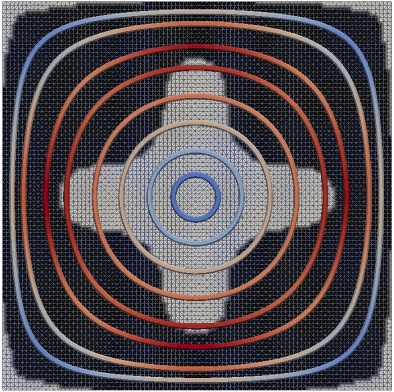
Mesh independence:

$1/h$	10	20	30	40	50	80
# it.	15	14	13	15	14	15

Reservoir flow

$$\Omega =]0,1[^2,\ g = 10,\ \mu = 1,\ \gamma = 10^4,\ f = 300(x_2 - 0.5, 0.5 - x_1)^T.$$





Conclusions

- Based on dual based regularization and a generalized differentiability notion, two Newton type algorithms for the solution of viscoplastic flow were constructed

Conclusions

- Based on dual based regularization and a generalized differentiability notion, two Newton type algorithms for the solution of viscoplastic flow were constructed
- Numerical algorithm I
  1. Global and local superlinear convergence in finite dimensions
  2. Determines active and inactive sets very accurately
  3. Can be used for the time-dependent convective problem in combination with BDF2

Conclusions

- Based on dual based regularization and a generalized differentiability notion, two Newton type algorithms for the solution of viscoplastic flow were constructed
- Numerical algorithm I
  1. Global and local superlinear convergence in finite dimensions
  2. Determines active and inactive sets very accurately
  3. Can be used for the time-dependent convective problem in combination with BDF2
- Numerical algorithm II
  1. Local superlinear convergence in infinite dimensions (mesh independence)
  2. No globalization needed

Conclusions

- Based on dual based regularization and a generalized differentiability notion, two Newton type algorithms for the solution of viscoplastic flow were constructed
- Numerical algorithm I
  1. Global and local superlinear convergence in finite dimensions
  2. Determines active and inactive sets very accurately
  3. Can be used for the time-dependent convective problem in combination with BDF2
- Numerical algorithm II
  1. Local superlinear convergence in infinite dimensions (mesh independence)
  2. No globalization needed
- Extension to other phenomena modeled by variational inequalities of the second kind.

Thank you!