A nonstandard phase field system of viscous Cahn-Hilliard type

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Joint work with

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The Cahn-Hilliard system $\partial_t \rho - \kappa \Delta \mu = 0 \,, \quad \mu = -\Delta \rho + f'(\rho)$

- aims to describe diffusion-driven phase-segregation processes in a two-phase material body;
- ρ, with $\rho(x,t) \in [0,1]$, is an *order-parameter* field interpreted as the scaled volumetric density of one of the two phases; $\kappa > 0$ is a *mobility* coefficient;
- μ is the *chemical potential*; f denotes a double-well potential with f' confined in $(0,1)$ and singular at endpoints.

The Cahn-Hilliard equation

$\boxed{\partial_t \rho = \kappa \Delta(-\Delta \rho + f'(\rho))}$ $\downarrow$ generalization by Fried and Gurtin

- *balance of contact and distance microforces*

$$\mathrm{div}\, \boldsymbol{\xi} + \pi + \gamma = 0$$

where $\boldsymbol{\xi}$ denotes the *microscopic stress* vector and the distance microforce is split in an internal part π and an external part γ;

- *balance law for the order parameter*

$$\partial_t \rho = -\, div\, \mathbf{h} + \sigma$$

where the pair $(\mathbf{h}, \sigma)$ is the *inflow* of ρ;

- *dissipation inequality* that accomodates diffusion

$$\partial_t \psi + (\pi - \mu)\partial_t \rho - \boldsymbol{\xi} \cdot \nabla(\partial_t \rho) + \mathbf{h} \cdot \nabla \mu \leq 0$$

for the *free energy density* ψ.

Set of constitutive prescriptions is acceptable

$$\psi = \widehat{\psi}(\rho, \nabla\rho), \quad \widehat{\pi}(\rho, \nabla\rho, \mu) = \mu - \partial_\rho \widehat{\psi}(\rho, \nabla\rho),$$
$$\widehat{\boldsymbol{\xi}}(\rho, \nabla\rho) = \partial_{\nabla\rho} \widehat{\psi}(\rho, \nabla\rho),$$

together with $\mathbf{h} = -\mathbf{M}\nabla\mu$, where $\mathbf{M} = \widehat{\mathbf{M}}(\rho, \nabla\rho, \mu, \nabla\mu)$; the tensor-valued *mobility mapping* **M** must satisfy the inequality

$$\boxed{\nabla\mu \cdot \widehat{\mathbf{M}}(\rho, \nabla\rho, \mu, \nabla\mu)\nabla\mu \geq 0}\,.$$

Then

$$\partial_t \rho = div\left(\mathbf{M}\nabla\left(\partial_\rho \widehat{\psi}(\rho, \nabla\rho) - div\left(\partial_{\nabla\rho}\widehat{\psi}(\rho, \nabla\rho)\right) - \gamma\right)\right) + \sigma$$

the Cahn-Hilliard equation is arrived at by taking

$$\widehat{\psi}(\rho, \nabla\rho) = f(\rho) + \frac{1}{2}|\nabla\rho|^2, \qquad \mathbf{M} = \kappa\mathbf{1}$$

both the microforce γ and the source σ identically null.

Podio-Guidugli proposed a modified version

- of Fried & Gurtin's derivation, where the order-parameter balance and dissipation inequality are both dropped and replaced by the *microenergy balance*

$$\partial_t \varepsilon = e + w, \quad e := -\,div\,\bar{\mathbf{h}} + \bar{\sigma}, \quad w := -\pi\,\partial_t\rho + \boldsymbol{\xi}\cdot\nabla(\partial_t\rho)$$

and the *microentropy imbalance*

$$\partial_t \eta \geq -\,div\,\mathbf{h} + \sigma, \quad \mathbf{h} := \mu\bar{\mathbf{h}}, \quad \sigma := \mu\,\bar{\sigma}.$$

- salient new feature of this approach → the *microentropy inflow* $(\mathbf{h}, \sigma)$ is deemed proportional to the *microenergy inflow* $(\bar{\mathbf{h}}, \bar{\sigma})$ through the *chemical potential* μ → a positive field.

Consistently, the free energy

is defined to be

$$\boxed{\psi := \varepsilon - \mu^{-1}\eta}$$

with the chemical potential playing the same role as *coldness* in the deduction of the heat equation.

- Just as absolute temperature can be seen as a macroscopic measure of microscopic *agitation*, its inverse - the coldness - measures microscopic *quiet*;

Consistently, the free energy

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$$\boxed{\psi := \varepsilon - \mu^{-1}\eta}$$

with the chemical potential playing the same role as *coldness* in the deduction of the heat equation.

- Just as absolute temperature can be seen as a macroscopic measure of microscopic *agitation*, its inverse - the coldness - measures microscopic *quiet*;
- likewise, the chemical potential can be seen as a macroscopic measure of microscopic *organization.*
- Combination of previous positions gives

$$\partial_t\psi \le -\eta\partial_t(\mu^{-1}) + \mu^{-1}\bar{\mathbf{h}}\cdot\nabla\mu - \pi\,\partial_t\rho + \boldsymbol{\xi}\cdot\nabla(\partial_t\rho),$$

an inequality that replaces the F&G one in restricting *à la* Coleman & Noll the possible constitutive choices.

What we get

- take all of the constitutive mappings delivering $\pi, \boldsymbol{\xi}, \eta$, and $\bar{\mathbf{h}}$ depending on the list $\rho, \nabla\rho, \mu, \nabla\mu$;
- choose $\boxed{\psi = \widehat{\psi}(\rho, \nabla\rho, \mu) = -\mu\, g(\rho) + f(\rho) + \frac{1}{2}|\nabla\rho|^2}$ with g nonnegative function on the domain of F

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- compatibility yields

$$\widehat{\pi}(\rho, \nabla\rho, \partial_t\rho, \mu) = \mu\, g'(\rho) - f'(\rho),$$

$$\widehat{\boldsymbol{\xi}}(\rho, \nabla\rho, \partial_t\rho, \mu) = \nabla\rho, \qquad \widehat{\eta}(\rho, \nabla\rho, \partial_t\rho, \mu) = -\mu^2\, g(\rho),$$

together with

$$\widehat{\bar{\mathbf{h}}}(\rho, \nabla\rho, \mu, \nabla\mu) = -\widehat{\mathbf{H}}(\rho, \nabla\rho, \mu, \nabla\mu)\nabla\mu, \quad \widehat{\mathbf{H}} = \kappa\mathbf{1}.$$

- constant and isotropic mobility $\kappa > 0$; assume that the external distance microforce γ and the source $\overline{\sigma}$ are null.

Nonlinear evolution system in the case $\boxed{g(\rho) = \rho}$

microforce balance $\quad div(\nabla\rho) + \mu - f'(\rho) = 0$

energy balance $\quad 2\rho\,\partial_t\mu + \mu\,\partial_t\rho - \kappa\Delta\mu = 0$

complemented with the homogeneous Neumann BC

$\partial_n\rho = \partial_n\mu = 0$ on the body's boundary and with the IC

$\rho|_{t=0} = \rho_0$ bounded away from 0 and 1, $\quad \mu|_{t=0} = \mu_0 \geq 0\,.$

First eq. $=$ same 'static' relation between μ and ρ as before.
Instead, second eq. is rather different for a number of reasons:

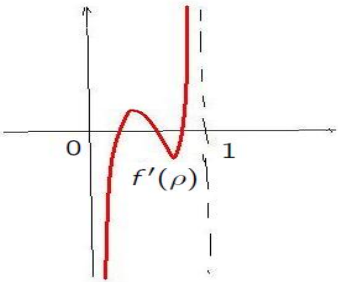
- is nonlinear (whereas $\partial_t\rho - \kappa\Delta\mu = 0$ is a linear equation);
- the time derivatives of ρ and μ are both present;
- nonconstant factors in front of both $\partial_t\mu$ and $\partial_t\rho$.

Sources of difficulties

the microforce balance is $$-\Delta\rho + f'(\rho) = g'(\rho)\mu$$

the energy balance reads $$2g(\rho)\mu_t + \mu g'(\rho)\rho_t - \kappa\Delta\mu = 0$$

- solution to the initial/boundary-value problem?
- is the density parameter ρ between 0 and 1?
- the chemical potential μ non-negative?



$\kappa > 0$ mobility coefficient,
($\kappa = 1$ in the sequel)
and
f = double well potential
f' singular at 0 and 1

Problem statement

Let $T > 0$, $\Omega \subset \mathbb{R}^3$ be bounded, open, with sufficiently smooth boundary Γ; $Q := \Omega \times (0, T)$, $\Sigma := \Gamma \times (0, T)$.

We consider:

$$2\, g(\rho)\, \mu_t + \mu\, g'(\rho)\, \rho_t - \Delta\mu = 0 \quad \text{in } Q \tag{1}$$

$$-\Delta\rho + f'(\rho) = \mu\, g'(\rho), \quad \text{in } Q \tag{2}$$

$$\frac{\partial\mu}{\partial n} = \frac{\partial\rho}{\partial n} = 0 \quad \text{on } \Sigma \tag{3}$$

$$\mu(0) = \mu_0 \geq 0\,, \quad \rho(0) = \rho_0 \in (0,1) \quad \text{in } \Omega \tag{4}$$

Here:

- ρ (order parameter, $\in [0,1]$) – volumetric density of one of the phases
- μ – chemical potential

Regularizing?

Let $T > 0$, $\Omega \subset \boldsymbol{R}^3$ be bounded, open, with sufficiently smooth boundary Γ; $Q := \Omega \times (0, T)$, $\Sigma := \Gamma \times (0, T)$.

We consider:

$$(\varepsilon + 2\,g(\rho))\mu_t + \mu\, g'(\rho)\, \rho_t - \Delta\mu = 0 \quad \text{in } Q$$

$$\delta\, \rho_t - \Delta\rho + f'(\rho) = g'(\mu) \quad \text{in } Q$$

$$\frac{\partial \mu}{\partial n} = \frac{\partial \rho}{\partial n} = 0 \quad \text{on } \Sigma$$

$$\mu(0) = \mu_0 \geq 0\,, \quad \rho(0) = \rho_0 \in (0, 1) \quad \text{in } \Omega$$

Here:

- $\varepsilon > 0$, $\delta > 0$ – regularization parameters

Why a fixed $\delta > 0$? Case $\mathbf{g}(\rho) = \rho$:

take $\rho_0 = 1/2$, μ_0 constant, and look for a space-independent solution. Then equations reduce to

$$\frac{d}{dt}\left((\varepsilon + 2\rho)^{1/2}\,\mu\right) = 0\,, \qquad f'(\rho) = \mu.$$

Hence,

$$\mu = z_0\,(\varepsilon + 2\rho)^{-1/2} \quad \text{and} \quad f'(\rho) = z_0\,(\varepsilon + 2\rho)^{-1/2}.$$

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Now, choose the potential f such that

$$f'(r) = z_0\,(\varepsilon + 2\,r)^{-1/2} \quad \text{for} \quad r \in I := [1/3, 2/3],$$

and pick any smooth/irregular $\rho : [0, T] \to I$ with $\rho(0) = 1/2$.

- We then get infinitely many smooth/irregular solutions !!!
- No uniqueness and no control on time regularity !!!

Assumptions

We introduce the spaces $H = L^2(\Omega)\,, \quad V = H^1(\Omega)\,,$
$W = \left\{ w \in H^2(\Omega)\,;\, \dfrac{\partial w}{\partial n} = 0 \text{ on } \Gamma \right\}.$

and postulate that

(A1) $f = f_1 + f_2,;\quad f_1 \in C^1(0,1)$ is convex;

$$\lim_{r\searrow 0} f_1'(r) = -\infty \quad \text{and} \quad \lim_{r\nearrow 1} f_1'(r) = +\infty;$$

$$g,\ f_2 \in C^2([0,1]);\quad g(r) \geq 0 \text{ for all } r \in [0,1];$$

(A2) $\mu_0 \in V\,;\quad \mu_0 \geq 0 \quad$ a. e. in $\quad \Omega;$

$$\rho_0 \in W;\quad 0 < \rho_0 < 1 \quad \text{in} \quad \Omega;\quad f'(\rho_0) \in H$$

$$(\Longrightarrow \quad \rho_0 \in C^0(\overline{\Omega})\,,\ f(\rho_0) \in H\,).$$

Existence and uniqueness

THEOREM 1: Let **(A1)**, **(A2)** be satisfied. Then (1)–(4) has a solution (μ, ρ) with:

$$\mu \in H^1(0,T;H) \cap L^2(0,T;W)\,;$$
$$\rho \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W)\,;$$
$$\mu \ge 0 \quad \text{a. e. in } Q\,; \qquad 0 < \rho < 1 \quad \text{a. e. in } Q\,;$$
$$f'(\rho) \in L^\infty(0,T;H)\,.$$

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THEOREM 2: Let, in addition,

(A3) $\mu_0 \in L^\infty(\Omega); \quad \inf_{x\in\Omega} \rho_0(x) > 0; \quad \sup_{x\in\Omega} \rho_0(x) < 1.$

Then the solution from **Theorem 1** is unique, and we have:

$$\mu \in L^\infty(Q); \quad \inf_Q \rho > 0; \quad \sup_Q \rho < 1.$$

THEOREM 3: Under the assumptions of **Theorem 2**, the ω-limit

$$\omega(\mu,\rho) \;=\; \{(\mu_\omega,\rho_\omega) : (\mu(t_n),\rho(t_n)) \to (\mu_\omega,\rho_\omega) \text{ weakly in } H\times V \text{ for a sequence } t_n \nearrow +\infty\}$$

is nonempty, as well as compact and connected in the topology of $H \times V$.

Large time behavior

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is nonempty, as well as compact and connected in the topology of $H\times V$. Moreover, every element $(\mu_\omega,\rho_\omega)\in\omega(\mu,\rho)$ is a **"'steady state"'**, i. e., μ_ω is a nonnegative constant, and ρ_ω satisfies

$$\rho_\omega \in W\,, \quad 0<\rho_\omega<1\,, \quad f'(\rho_\omega)\in H\,,$$

and

$$-\Delta\rho_\omega + f'(\rho_\omega) = \mu_\omega g'(\rho_\omega) \quad \text{a. e. in } \Omega\,.$$

Outline of the existence proof

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1. Approximation: Introduce a delay in (2):

$$\delta \rho_t - \Delta\rho + f'(\rho) = (\mathcal{T}_\tau \mu) g'(\rho),$$

where, for $\tau \in (0, T)$,

$$(\mathcal{T}_\tau \mu)(t) = \begin{cases} \mu(t-\tau) & , \quad t \geq \tau \\ \mu_0 & , \quad 0 \leq t < \tau \end{cases}$$

For every $\tau > 0$, one obtains a unique solution (μ^τ, ρ^τ) to (1), $(2)_\tau$, (3), (4) with the regularity as in **Theorem 1**.

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2. A priori estimates: See below.

3. Passage to the limit as $\tau \searrow 0$: by different compactness results and monotonicity arguments (for $f_1'(\rho)$)

A priori estimates (τ omitted)

- Test (1) by μ and use the identity

$$(\varepsilon\,\mu_t + 2\,g(\rho)\,\mu_t + \mu\,g'(\rho)\,\rho_t)\,\mu = \left(\left(\frac{\varepsilon}{2} + g(\rho)\right)\mu^2\right)_t$$

$\Longrightarrow$

$$\int_\Omega \left(\frac{\varepsilon}{2}\,\mu^2 + g(\rho)\,\mu^2\right)(t)\,dx + \int_0^t\int_\Omega |\nabla\mu|^2\,dx\,ds = C_0\,.$$

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- Remark: Testing by $-\mu^- = \max\{-\mu, 0\}$ leads to $\mu^- = 0$, hence $\mu \geq 0$.
- Testing of (2) by ρ_t and by $-\Delta\rho$ yields:

$$\|\rho\|_{H^1(0,T;H)\cap L^2(0,T;W)} + \|f_1'(\rho)\|_{L^2(Q)} \leq C\,.$$

A priori estimates II

- Differentiate (2) with respect to t and test by ρ_t $\Longrightarrow$

$$\frac{\delta}{2}\int_\Omega |\rho_t(t)|^2\,dx + \int_0^t\!\!\int_\Omega |\nabla\rho_t|^2\,dx\,ds$$

$$\leq \int_0^t\!\!\int_\Omega C(1+|\mathcal{T}_\tau\mu|)|\rho_t|^2\,dx\,ds + \int_0^t\!\!\int_\Omega \partial_t(\mathcal{T}_\tau\mu)\,g'(\rho)\,\rho_t\,dx\,ds\,.$$

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- Now, substitute $\mu_t = (\varepsilon + 2g(\rho))^{-1}(\Delta\mu - \mu\,g'(\rho)\,\rho_t)$, integrate by parts, and estimate the resulting terms with the help of Hölder, Gronwall, ... obtaining

$$\|\rho_t\|_{L^\infty(0,T;H)} + \|\rho\|_{H^1(0,T;V)} \leq C\,.$$

A priori estimates III

- Test (2) by $-\Delta\rho$ and by $f_1'(\rho)$ $\Longrightarrow$

$$\|\rho\|_{L^\infty(0,T;W)} + \|f_1'(\rho)\|_{L^\infty(0,T;H)} \leq C\,.$$

A priori estimates III

- Test (2) by $-\Delta\rho$ and by $f_1'(\rho)$ $\Longrightarrow$

$$\|\rho\|_{L^\infty(0,T;W)} + \|f_1'(\rho)\|_{L^\infty(0,T;H)} \leq C\,.$$

- Add μ on both sides of (1) and test by μ_t $\Longrightarrow$

$$\frac{\varepsilon}{2}\int_0^t\int_\Omega \mu_t^2\,dx\,ds + \frac{1}{2}\|\mu(t)\|_V^2 \leq \ldots + \underbrace{C\int_0^t\int_\Omega |\mu|\,|\rho_t|\,|\mu_t|\,dx\,ds}_{=:\ I}\,.$$

We have

$$I \leq \frac{\varepsilon}{4}\int_0^t\int_\Omega \mu_t^2\,dx\,ds + \frac{C}{\varepsilon}\int_0^t \|\rho_t(s)\|_V^2\,\|\mu(s)\|_V^2\,ds$$

$\overset{\text{GRONWALL}}{\Longrightarrow}$

$$\|\mu\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \leq C$$

Passage to the limit as $\boldsymbol{\tau \searrow 0}$

For a subsequence $\tau_n \searrow 0$, we have:

$\mu_{\tau_n} \rightharpoonup^* \mu$ in $H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$

$\rho_{\tau_n} \rightharpoonup^* \rho$ in $W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W)$

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Passage to the limit as $\tau \searrow 0$

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This implies strong convergences for μ_{τ_n} and ρ_{τ_n}, and

$f_1'(\rho_{\tau_n}) \to f_1'(\rho)$ weakly* in $L^\infty(0,T;H)$ (monotonicity !)

$g(\rho_{\tau_n})\,\partial_t\mu_{\tau_n} \to g(\rho)\,\mu_t$ weakly in $L^2(0,T;L^{3/2}(\Omega))$

$\mu_{\tau_n}\,g'(\rho_{\tau_n})\,\partial_t\rho_{\tau_n} \to \mu\,g'(\rho)\,\rho_t$ weakly in $L^1(0,T;H)$

$\Longrightarrow$ (μ,ρ) is a solution, since the conditions $\mu \ge 0$ and $0 < \rho < 1$ follow from pointwise a. e. convergence.

$\Longrightarrow$ **Theorem 1** is proved!

Boundedness of μ and of $\mathbf{f}'(\rho)$

1. Now assume that $\mu_0 \in L^\infty(\Omega)$. We make use of the following result:

If $S_{j+1} \leq C\, 2^j\, S_j^p$ with $p > 1$ and $S_0 \ll 1$, then $S_j \to 0$.

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In a very technical proof it is shown that the property holds with the choices

$$S_j := \|\chi_{\{\mu > k_j\}}\|_{L^2(0,T;L^4(\Omega))}\,, \quad p := \frac{8}{7}\,,$$

for a suitably chosen sequence $\{k_j\} \nearrow k_\infty < +\infty$.

$$\Longrightarrow \quad \|\mu\|_{L^\infty(Q)} \leq C\,.$$

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for a suitably chosen sequence $\{k_j\} \nearrow k_\infty < +\infty$.

$$\Longrightarrow \quad \|\mu\|_{L^\infty(Q)} \le C.$$

2. As now $\mu \in L^\infty(Q)$, testing (2) by standard test functions leads to

$$0 < \rho_* \le \rho \le \rho^* < 1 \quad \text{in } Q,$$

for suitable ρ_*, ρ^*. Here, **(A1)** is used.

Asymptotic behavior as $\varepsilon \searrow 0$ $\boxed{\text{Case } \mathbf{g}(\boldsymbol{\rho}) = \boldsymbol{\rho}}$

We rewrite the system in the form:

$$(\varepsilon + 2\,\rho^\varepsilon)\,\mu_t^\varepsilon + \mu^\varepsilon\,\rho_t^\varepsilon - \Delta\mu^\varepsilon = 0$$

$$\delta\,\rho_t^\varepsilon - \Delta\rho^\varepsilon + f'(\rho^\varepsilon) = \mu^\varepsilon$$

with initial and boundary conditions.

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Idea: Write first eq. in the form (little miracle)

$$(\varepsilon\,\mu^\varepsilon + 2\,\mu^\varepsilon\,\rho^\varepsilon)_t - \Delta\mu^\varepsilon = \mu^\varepsilon\,\rho_t^\varepsilon$$

with the aim to obtain in the limit:

$$(2\,\mu\,\rho)_t - \Delta\mu = \mu\,\rho_t\,.$$

Asymptotic behavior as $\varepsilon \searrow 0$ Case $\mathbf{g}(\rho) = \rho$

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Problem: For $\varepsilon \searrow 0$, we do not have any estimate for μ^ε_t.

Idea: Write first eq. in the form (little miracle)

$$(\varepsilon\,\mu^\varepsilon + 2\,\mu^\varepsilon\,\rho^\varepsilon)_t - \Delta\mu^\varepsilon = \mu^\varepsilon\,\rho^\varepsilon_t$$

with the aim to obtain in the limit:

$$(2\,\mu\,\rho)_t - \Delta\mu = \mu\,\rho_t\,.$$

Result:

- Last eq. is meaningful: it turns out that ρ_t and $(\mu\,\rho)_t$ exist, while μ_t may not exist
  ($\Longrightarrow$ $(\mu\,\rho)_t$ cannot be evaluated using the product rule!)

Convergence

THEOREM 4: Let **(A1)**, **(A2)** be satisfied. Then there exist a sequence $\varepsilon_n \searrow 0$ and functions (μ, ρ) such that

$$\mu_{\varepsilon_n} \to \mu \quad \text{weakly* in} \quad L^\infty(0,T;H) \cap L^2(0,T;V)$$

$$\rho_{\varepsilon_n} \to \rho \quad \text{weakly* in} \quad H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$$

$$\mu \geq 0 \quad \text{and} \quad 0 < \rho < 1 \quad \text{a.e. in } Q$$

$$\mu\,\rho \in W^{1,5/4}(0,T;V^*) \quad \text{and} \quad f'(\rho) \in L^2(Q)$$

Moreover, we have for all $v \in V$ and a.e. in $(0,T)$

$$2\langle(\mu\,\rho)_t, v\rangle_{V^*,V} + \int_\Omega \nabla\mu(t)\cdot\nabla v\,dx = \int_\Omega \mu(t)\,\rho_t(t)\,v\,dx\,,$$

and $\delta\rho_t - \Delta\rho + f'(\rho) = \mu$ a.e. in Q,

$$(\mu\,\rho)(0) = \mu_0\,\rho_0\,, \quad \rho(0) = \rho_0\,, \quad \text{a.e. in } \Omega.$$

Convergence II

Problem: Find a priori estimates independent of ε !

Results:

- It can be shown that $\rho^\varepsilon \geq \rho_* > 0$ in Q $\forall\, \varepsilon > 0$.
- By this, we check that

$$\|\mu^\varepsilon\|_{L^\infty(0,T;H)} + \|\mu^\varepsilon\|_{L^2(0,T;V)} \leq C\,.$$

- Testing (2) by ρ^ε_t implies, by virtue of the "'little miracle"',

$$\delta \int_0^t \int_\Omega |\rho^\varepsilon_t|^2\, dx\, ds + \left[\frac{1}{2} \|\nabla \rho^\varepsilon(t)\|_H^2 + \int_\Omega f(\rho^\varepsilon(t))\, dx \right]_0^t$$

$$= \int_0^t \int_\Omega \mu^\varepsilon\, \rho^\varepsilon_t\, dx\, ds = \int_\Omega (\varepsilon\, \mu^\varepsilon(t) + 2\, \mu^\varepsilon(t)\, \rho^\varepsilon(t))\, dx - C$$

$$\Longrightarrow \qquad \|\rho^\varepsilon_t\|_{L^2(Q)} + \|\rho^\varepsilon\|_{L^\infty(0,T;V)} \leq C$$

Convergence III

- Testing by $f_1'(\rho^\varepsilon)$ and by $-\Delta\rho^\varepsilon$ yields that

$$\|f_1'(\rho^\varepsilon)\|_{L^2(Q)} + \|\rho^\varepsilon\|_{L^2(0,T;W)} \leq C$$

Conclusions:

- There are (μ, ρ, φ) such that (at least for a subsequence $\varepsilon_n \searrow 0$) we pass to the limit, $f_1'(\rho_{\varepsilon_n}) \to \varphi$ weakly in $L^2(Q)$, $\mu \geq 0$ and $\rho \geq \rho_* > 0$ a. e. in Q.
- In view of the compact embedding $V \subset L^p(\Omega)$ for $1 \leq p < 6$, we also have

$$\rho_\varepsilon \to \rho \quad \text{strongly in} \quad C^0([0,T]; L^p(\Omega)) \quad \text{for} \quad p < 6,$$

and a monotonicity argument for f_1' yields $\varphi = f_1'(\rho)$. In summary,

$$\delta\rho_t - \Delta\rho + f_1'(\rho) = \mu.$$

Convergence IV

Next, we have for every $\quad v \in L^5(0,T;V)$:

$$\left| \int_0^T \int_\Omega \mu\, \rho_t\, v\, dx\, ds \right| \le \|\mu\|_{L^{10/3}(Q)} \|\rho_t\|_{L^2(Q)} \|v\|_{L^5(Q)}$$

$$\le C \|v\|_{L^5(0,T;V)} ,$$

owing to the continuity of the embedding
$L^\infty(0,T;H) \cap L^2(0,T;V) \subset L^{10/3}(Q)$.

$\Longrightarrow \quad \|u_t^\varepsilon\|_{L^{5/4}(0,T;V^*)} \le C, \qquad$ for $\quad u^\varepsilon := \varepsilon\, \mu^\varepsilon + 2\, \mu^\varepsilon\, \rho^\varepsilon$.

Now: Strong convergence of ρ_ε (with $p = 4$) and $\quad \mu_\varepsilon \to \mu$ weakly in $\quad L^2(0,T;L^4(\Omega)) \quad$ imply that

$$\mu^\varepsilon \rho^\varepsilon \to \mu\, \rho \quad \text{weakly in} \quad L^2(0,T;H)$$

$$\Longrightarrow \quad u^\varepsilon \to 2\, \mu\, \rho \quad \text{weakly in} \quad L^2(0,T;H) \cap W^{1,5/4}(0,T;V^*) .$$

Convergence V

Thus $u^\varepsilon \to 2\,\mu\,\rho$ weakly in $C^0([0,T];V^*)$ $\Longrightarrow$
$u^\varepsilon(0) = \varepsilon\,\mu_0 + 2\,\mu_0\,\rho_0 \to (2\,\mu\,\rho)(0)$ weakly in V^*,
so that $(\mu\,\rho)(0) = \mu_0\,\rho_0$.

Besides, it can be shown that $\|u^\varepsilon\|_{L^2(0,T;W^{1,3/2}(\Omega))} \leq C$
Aubin-Lions lemma $\Longrightarrow$

$u_\varepsilon \to 2\,\mu\,\rho$ strongly in $L^2(0,T;L^q(\Omega))$ for $1 \leq q < 3$.

Lemma: It holds $\|\mu^\varepsilon - \mu\|_{L^2(Q)} \to 0$.

Consequence: $\mu^\varepsilon\,\rho_t^\varepsilon \to \mu\,\rho_t$ weakly in $L^1(Q)$.
$\Longrightarrow$ the limit procedure is complete!

Remarks and possible extensions

1. Also for $\varepsilon = 0$ the $\omega-$limit $\omega(\mu, \rho)$ is nonempty and consists of steady states.
2. More general forms of the potential $f(\rho)$ can be treated (to allow f_1' be any maximal monotone graph).
3. Another ad hoc uniqueness proof is available. Optimal control problems, for distributed and boundary controls, have been investigated.
4. Mobility coefficient κ may be nonlinear function of μ and possibly of ρ too. Instead, what about nonlocal models?
5. Numerical approximation? starting from time discretization ...
6. It should be possible to include the case of vectorial order parameters.

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Many thanks
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