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Motion of Regular Closed Curves by Singular Weighted Mean Curvature

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Outline:

– Introduction

– κ_γ and the equation $\beta V = \sigma + \kappa_\gamma$ in local coordinates

– Construction of interfacial curves

– Regular variational solutions

– Uniqueness(?)

– Viscosity solutions for evolving graphs

1. Introduction

The Gibbs-Thomson relation on the evolving surface of a crystal reads,

$$\beta V = \sigma + \kappa_\gamma \text{ on } \Gamma(t). \tag{1}$$

We want:

1) to study evolution of closed planar curves driven by this eq.
2) to look at (1) from the view point of parabolic problems.

History:

J.Taylor, Angenent – Gurtin

Andreu, Bellettini, Caselles, Fukui, M.-H.Giga, Y.Giga, Mazon, Novaga, Paolini, PR

more recently: Mucha-PR, Bonforte-Figalli

2. The equation in a local coordinate system

Here, we shall construct variational solutions to a ‘geometric’ evolution problem. On the way we will keep track of its parabolic nature, which we hope to exploit eventually. We will not consider the problem in its full generality but rather what is feasible.

2.1. Ingredients of (1)

a) $\Gamma(t) \subset \mathbb{R}^2$ – a closed curve, (here: a *bent rectangle*), $\mathbf{n}$ its outer normal.

b) Formally, the weighted mean curvature is $\kappa_\gamma = -\text{div}_S \left(\nabla_\zeta \gamma(\zeta)|_{\zeta=\mathbf{n}(x)} \right)$.
If $\gamma(\zeta) = |\zeta|$, then κ_γ is the Euclidean mean curvature. Here,

$$\gamma(p_1, p_2) = |p_1|\gamma_\Lambda + |p_2|\gamma_R, \tag{2}$$

c) $\beta = \beta(\mathbf{n})$ – a kinetic coefficient.

d) σ – the driving (supersaturation, temperature, pressure, ...). It satisfies:

$$\sigma(x_1, x_2) = \sigma(\pm x_1, \pm x_2) \tag{3}$$

and the *Berg's effect*, i.e.

$$x_i \frac{\partial \sigma}{\partial x_i}(x_1, x_2) > 0 \quad x_i \neq 0,\ i = 1, 2. \tag{4}$$

The surface energy $\int_\Gamma \gamma(\mathbf{n})\, d\mathcal{H}^1$ under the volume constraint is minimized by a scaled Wulff shape, W_γ, i.e. a ball in the space dual to $(\mathbb{R}^2, \gamma)$. For γ given by (2) the Wulff shape is a rectangle. Our curves are special perturbations of W_γ, *bent rectangles*,

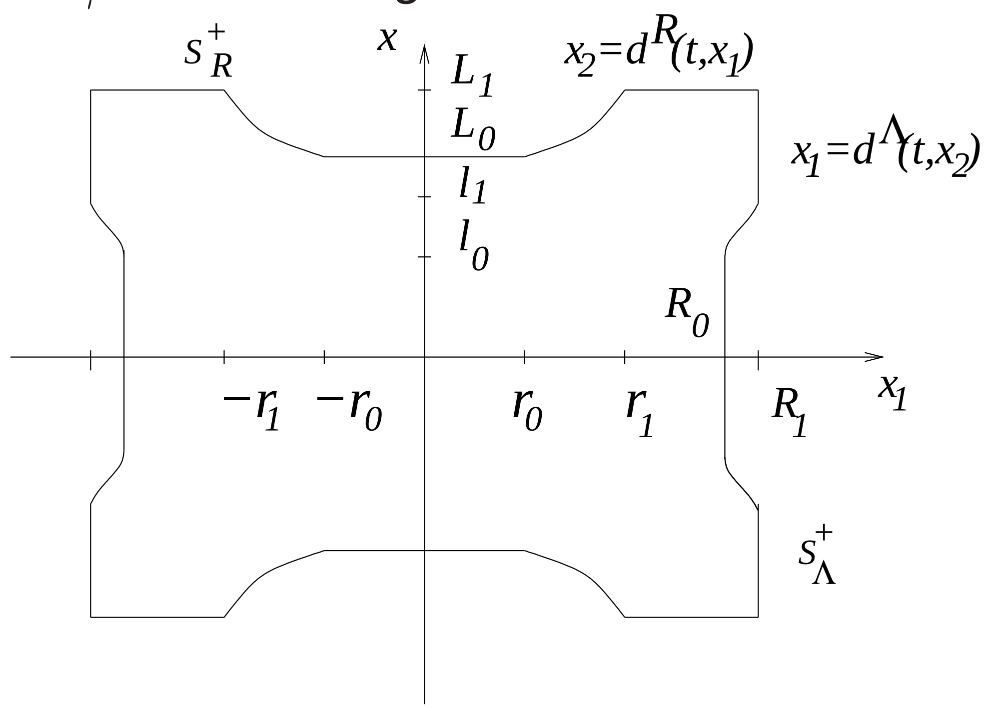


Fig. 1

We will consider evolution of *regular bent rectangles*. Each side of a bent rectangle is a graph of a Lipschitz function such that it has three facets. For us regularity means that

$$d^{\Lambda}(t,\cdot) \in C^2([l_0(t), l_1(t)]) \quad \text{(resp. } d^R(t,\cdot) \in C^2([r_0(t), r_1(t)])).$$

2.2 The weighted mean curvature κ_γ

The main difficulty is how to interpret

$$\sigma - \mathrm{div}_S \left(\nabla_\zeta \gamma(\zeta)|_{\zeta=\mathbf{n}(x)} \right). \tag{5}$$

[After jumping forward to the local coordinates (5) becomes

$$\sigma - W_p(d_x)_x.] \tag{6}$$

γ (resp. W) is convex, hence a.e. differentiable, but not at the normal vectors to the facets.

Ways to interpret (5):

1) γ is convex, so $\partial\gamma$ is always well-defined, but we need to find a selection ξ of $\partial\gamma$. Hence, (5) becomes

$$\sigma - \frac{\partial \xi}{\partial x}$$

We will use that (5) is the E-L of functionals

$$\mathcal{E}_\Lambda(\xi) = \int_{S_\Lambda} |\sigma - \text{div}_S \xi|^2 \mathcal{H}^1, \qquad \mathcal{E}_R(\xi) = \int_{S_R} |\sigma - \text{div}_S \xi|^2 \mathcal{H}^1. \quad (7)$$

2) the method of viscosity solutions, developed by M.-H.Giga and Y.Giga for graphs, also requires solving an obstacle problem to give meaning to (6): $\sigma - W_p(d_x)_x = \frac{d\Lambda}{dx}$, where Λ is a minimizer of a variational problem.

3) as the divergence of a special composition of multivalued functions, developed by P.B.Mucha and PR, (PBM gave a talk on this on Monday).

We follow 1). After constructing solutions we will discuss 2).

Variational solutions

A family of couples $(\Gamma(t), \xi(t))_{t\geq 0}$ will be called a *variational solution* iff $\Gamma(t)$ is a bent rectangle and $\xi(t)$ is a solution to

$$\min\{\mathcal{E}_\Lambda(\xi) : \text{div}_S \xi \in L^2,\ \xi(x) \in \partial\gamma(\mathbf{n}(x))\},$$
$$\min\{\mathcal{E}_R(\xi) : \text{div}_S \xi \in L^2,\ \xi(x) \in \partial\gamma(\mathbf{n}(x))\}$$

and eq. (1) is satisfied in the L^2 sense.

An advantage of variational solutions is that they are 'explicit' compared to viscosity solutions. However, 3) also yields explicit solutions. Presently, methods 2) and 3) cannot be applied to construct evolution of closed curves while 1) works.

2.3 Equation (1) in a local coordinate system

If we adopt notation as in Fig. 1 then, equation (1) takes the following form

$$\beta(d_x^\Lambda)d_t^\Lambda = \frac{\partial \xi^\Lambda}{\partial x} \quad \text{for} \quad s \in (-L_1, L_1)$$
$$\beta(d_x^R)d_t^R = \frac{\partial \xi^R}{\partial x} \quad \text{for} \quad s \in (-R_1, R_1), \tag{8}$$

where ξ^Λ, ξ^R are minimizers of $\mathcal{E}_\Lambda$, $\mathcal{E}_R$.

Proposition 2.1 Taking into account the form of the minimizers (8) becomes

$$
\begin{aligned}
\dot{R}_0/m(0) &= \int_0^{l_0} \sigma(t, R_0, s)\, ds + \frac{\gamma_R}{l_0} && \text{on} \quad [0, l_0] \\
d_t^\Lambda &= \sigma(t, d^\Lambda, s) m(d_x^\Lambda) && \text{for} \quad s \in (l_0, l_1) \\
\dot{R}_1/m(0) &= \int_{l_1}^{L_1} \sigma(t, R_1, s)\, ds + \frac{2\gamma_R}{L_1 - l_1} && \text{on} \quad [l_1, L_1] \\
\dot{L}_0/m(0) &= \int_0^{r_0} \sigma(t, s, L_0)\, ds + \frac{\gamma_\Lambda}{r_0} && \text{on} \quad [0, r_0] \\
d_t^R &= \sigma(t, s, d^R) m(d_x^R) && \text{for} \quad s \in (r_0, r_1) \\
\dot{L}_1/m(0) &= \int_{r_1}^{R_1} \sigma(t, s, L_1)\, ds + \frac{2\gamma_\Lambda}{R_1 - r_1} && \text{on} \quad [r_1, R_1].
\end{aligned}
\tag{9}
$$

Here, $m(d_x) = 1/\beta(d_x)$.

System (9) is not closed until we specify evolution of $r_0(\cdot)$, $r_1(\cdot)$, $l_0(\cdot)$, $l_1(\cdot)$, these are genuine free boundaries. Once we know the position of the interface this system can be viewed as a system of Hamilton-Jacobi equations.

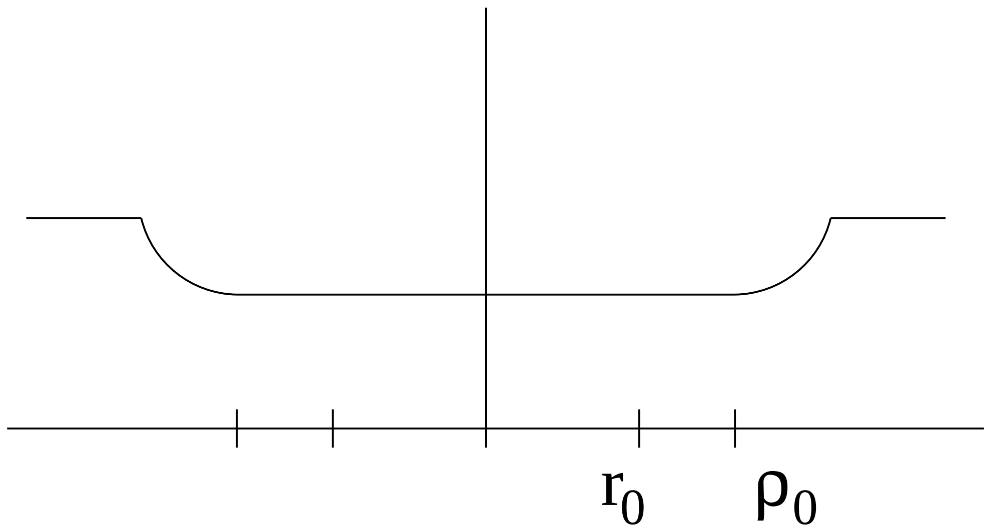
At the interface r_i, (resp. l_i), $i = 0, 1$, the *matching condition*

$$L_i(t) = d^R(t, r_i), \qquad \text{(resp. } R_i(t) = d^\Lambda(t, l_i)) \tag{10}$$

is equivalent to continuity of d^R (resp. d^Λ).

General observation for r_0

- characteristics of the HJ eq. turn left, $\dot{x}(t,\zeta) = -\sigma(t,x,d^R)m'(d^R_x) < 0$;
- r_0 it is defined as a boundary of the coincidence set.



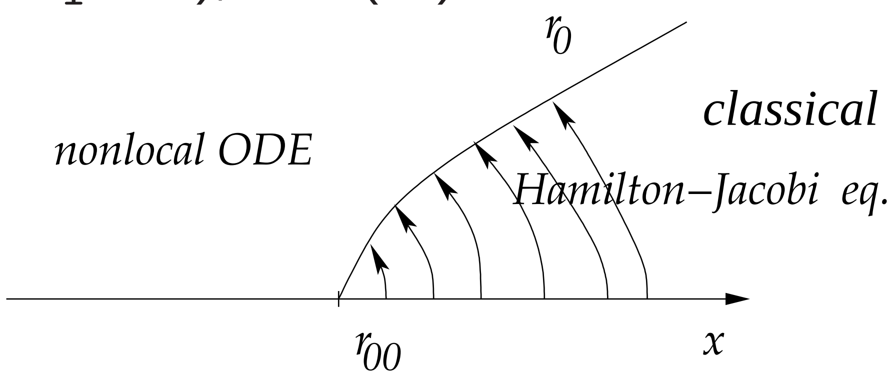
Functional $\mathcal{E}_R$ is minimized over a closed convex set, with the constraint $\xi(x) \in \partial\gamma(\mathbf{n}(x))$. In general, we have a nontrivial coincidence set on $S_R^{\pm}$,

$$\{x:\ \xi(x) = (\pm\gamma_\Lambda, \gamma_R)\} = [-\rho_0, r_0] \cup [r_0, \rho_0].$$

- in a generic case $\dot{r}_0 < 0$ or $\dot{r}_0 > 0$.

Specific observation for r_0 (resp. r_1):

- if $\dot{r}_0 > 0$ (resp. $\dot{r}_1 < 0$), then (10) defines a 'shock wave'



- if $\dot{r}_0 < 0$, (resp. $\dot{r}_1 > 0$) and if $d_x(\cdot, r_0) = 0$ (resp. $d_x(\cdot, r_1) = 0$), then, ξ the solution to the obstacle problem (7) meets $\partial\gamma(\mathbf{n})$ tangentially (Kinderlehrer, Stampacchia),

$$\frac{\partial \xi}{\partial x_1}(r_0) = 0 \qquad \text{(resp. } \frac{\partial \xi}{\partial x_1}(r_1) = 0\text{).} \tag{11}$$

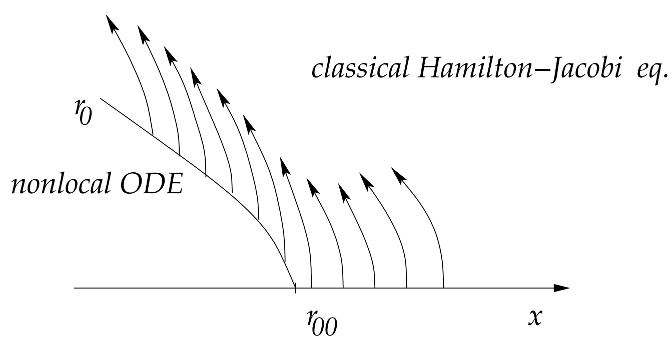
We call (11) the *tangency condition* (TC).

One can show that (11) is equivalent to

$$\sigma(t, r_0, L_0) = \fint_0^{r_0} \sigma(t, s, L_0)\, ds + \frac{\gamma_\Lambda}{r_0},$$
$$\text{(resp. } \sigma(t, r_1, L_1) = \fint_{r_1}^{R_1} \sigma(t, s, L_1)\, ds - \frac{2\gamma_\Lambda}{R_1 - r_1}\text{)}. \qquad (12)$$

The HJ with boundary data on r_0 can be solved iff r_0 is faster than the characteristics, i.e.

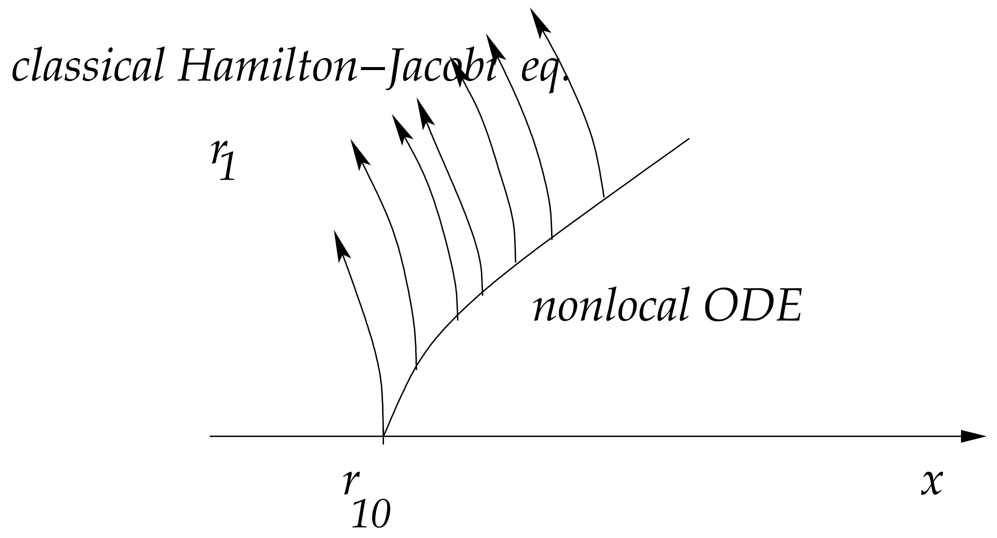
$$\dot{r}_0(t) < \dot{x}(t, r_0(t)).$$



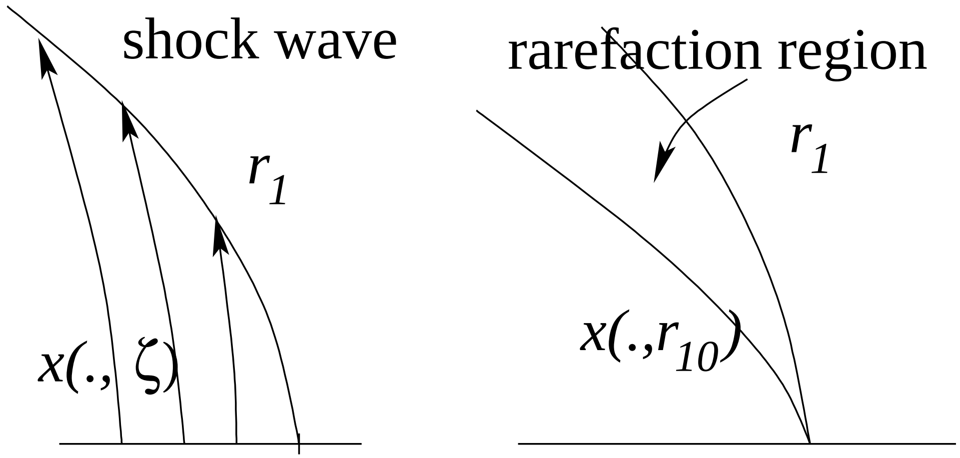
The specific analysis of r_1 is slightly different.
If $\dot{r}_1 > 0$ (i.e. r_1 is a tangency curve), the problem

$$d_t = \sigma(t, x, d)m(d_x), \qquad d(t, r_1(t)) = L_1(t)$$

can always be solved,



If $\dot{r}_1 < 0$ (i.e. it is a matching curve), we could have two possibilities,



The shock wave case occurs, iff

$$\frac{m(p_0)}{p_0}\sigma(t,x,d_0) - m(0)\frac{\left(\int_{r_1}^{R_1}\sigma(t,s,d_0)\,ds - 2\gamma_\Lambda\right)}{p_0(R_1 - r_1)} < -m_p(p_0)\sigma(t,x,d_0). \tag{13}$$

We do not know how to handle the rarefaction region, yet.

3. Construction of the interfaces

Proposition 3.1

(a) $\dot{r}_0 < 0$ (resp. $\dot{r}_1 > 0$) iff r_0 (resp. r_1) is a tangency curve, i.e. (12) holds for all $t \geq 0$. Tangency curves are uniquely constructed by the Implicit Function Theorem.

(b) If $\dot{r}_0 > 0$ (resp. $\dot{r}_1 < 0$), then (12) does not hold and the curve is defined solely by the matching condition, eq. (10), i.e.

$$L_0(t) = d^R(t, r_0), \qquad \text{(resp. } L_1(t) = d^R(t, r_1))$$

equivalent to continuity of d^R.

If $d_x^+(0, r_0(0)) > 0$ (resp. ${}_x^+(0, r_1(0)) > 0$), then the interfacial curve is unique.

We can handle the whole system,

Theorem 3.2 Let us suppose that $\Gamma(0)$ is such that $d_0^\Lambda \in C^2([l_0, l_1])$, $d_0^R \in C^2([r_0, r_1])$, σ satisfies (3), (4). The red provisions hold.

We assume that one of the following conditions occurs at each interfacial point r_i, l_i, $i = 0, 1$.

(a) the TC holds at r_0 (resp. l_0) and $\dot{r}_0 < 0$ (resp. $\dot{l}_0 < 0$), i.e. the facet shrinks

or (b) $\dot{r}_0 > 0$ (resp. $\dot{l}_0 > 0$) and $d_x^R(r_0(0)) > 0$ (resp. $d_x^\Lambda(l_0(0)) > 0$), i.e. the facet shrinks. (Similar conditions at r_1 and l_1).

Then, there exists a variational solution to (1).

Idea: we are basically done once we constructed the interfacial curves. In the cases specified above we can solve the HJ and the system. Finding the Cahn-Hoffman vector ξ is easy, a formula pops up from the minimization process.

4. Uniqueness

Theorem 4.1 If $(\Gamma^i(t), \xi^i(t))$, $i = 1, 2$, are two regular variational solutions to (1) with the i.c. Γ_0, whose all interfacial curves are tangency curves, then $(\Gamma^1(t), \xi^1(t)) = (\Gamma^2(t), \xi^2(t))$ for all $t \geq 0$.

Idea of the proof. If we have two regular variational solutions d^1, d^2, then in a local coordinate system we can consider their difference $p = d^2 - d^1$. This is a solution to the following problem,

$$p_t = Ap_x + Bp - \left(\frac{\partial \xi^2}{\partial x} - \frac{\partial \xi^1}{\partial x}\right),$$

where

$$A = \sigma^2 \frac{m(d_x^2) - m(d_x^1)}{d_x^2 - d_x^1}, \quad B = m(d_x^1)\frac{\sigma^2 - \sigma^1}{d^2 - d^1}.$$

Since the last term has a sign due to monotonicity of ξ and regularity of solutions imply boundedness of A_x we may apply Gronwall inequality to deduce that $p \equiv 0$.

5. Observations:

– We constructed variational solutions only for regular bent rectangles, but not for all configurations, see the red text provisions.

– The angle at the corner is artificially set to be right;

– We did not fully exploit the parabolic nature of eq. (8), i.e.

$$d_t = \frac{1}{\beta(d_x)}\frac{\partial \xi}{\partial x} \qquad x \in (-L, L),$$

where $\xi \in \partial\gamma(\mathbf{n})$. More precisely, we have on each side of $\Gamma(t)$

$$d_t = a(d_x)\left((W'(d_x))_x + \sigma(t,x,d)\right),$$

where W is merely convex, e.g. $W(p) = |p|$.

Having more tasks in sight we need a better suited tool. We have seen that hand constructing variational has some limitations.

6. Viscosity solutions for graphs (Giga-Giga)

Let us consider evolution of a graph of u on $\mathbb{R}$ by

$$\begin{aligned} &d_t = a(d_x)\left((W'(d_x))_x + \sigma(t,x)\right) \qquad \text{in } \mathbb{R} \\ &d(0,x) = d_0(x). \end{aligned} \tag{14}$$

This eq. resembles eq. (1) in the local coordinates.

In order to define *viscosity solutions* we have to determine $(W'(\phi_x))_x + \sigma(t,x)$ for a ‘smooth’ test function ϕ. Here, we consider $W(p) = |p|$.

Definition 6.1 Admissible test functions are $\phi(x,t) = f(x) + g(t)$, where $f, g \in C^2$ and if $f'(x_0) = 0$, then $f'|_I = 0$, where I is an open interval, $x_0 \in I$.

Suppose $\phi = f + g$ is admissible, $I = (a,b)$ is a connected component of $f' = 0$. We set

$$(W'(d_x))_x + \sigma(t,x) = \Lambda^{\sigma}_{\chi_l \chi_r}(x, I) := \frac{d\xi}{dx}$$

where ξ is a unique solution to

$$\min\{\int_I |\frac{d\xi}{dx}|^2 : \ \xi \in K^Z_{\chi_l \chi_r}\}$$

and $Z(x) = \int_0^x \sigma(s,t)\, ds$.

Here χ_l, $\chi_r = \pm 1$ and $K^Z_{\chi_l \chi_r}$ consists of such $H^1(I)$ functions ω, that

$$Z(x) - \gamma_\Lambda \leq \omega(x) \leq Z(x) + \gamma_\Lambda \; x \in I$$

and

$$\omega(a) = Z(a) - \chi_l \gamma_\Lambda, \quad \omega(b) = Z(b) + \chi_r \gamma_\Lambda.$$

Definition 6.2 A continuous real-valued function u on $Q_T := (0, T) \times (a, b)$ is a (viscosity) *subsolution* of (14) in Q_T if for each $(\hat{t}, \hat{x}) \in Q_T$

$$\psi_t(\hat{t}, \hat{x}) + a(\psi_x(\hat{t}, \hat{x}))\Lambda^{Z(\hat{t}, \cdot)}(\psi(\hat{t}))\,(\hat{x}) \leq 0, \tag{15}$$

here $\psi(\hat{t}) := \psi(\hat{t}, \cdot)$ and ψ is an admissible test function fulfilling

$$\max_Q (u - \psi) = (u - \psi)\,(\hat{t}, \hat{x}). \tag{16}$$

A continuous (viscosity) *supersolution* is defined by replacing max by min in (16) and the inequality (15) by the opposite one. If u is both a sub- and supersolution, u is called a *viscosity solution*

Theorem 6.1. (Comparison Principle, Giga-Giga-PR)
Let u and v be respectively continuous sub- and supersolutions of (14) in Q_T, where (a, b) is a bounded open interval. If $u \leq v$ on the parabolic boundary $\partial_p Q_T (= [0, T) \times \{a, b\} \cup \{0\} \times [a, b])$ of Q_T, then $u \leq v$ in Q_T.

Theorem 6.2 There exists a variational solution of (14).

Theorem 6.3 Variational solutions of (14) are viscosity solutions, hence they are unique.

Question: Can we apply the viscosity theory to bent rectangles?