Evolution of domains

by spectral flows

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research on:

"Shape flows for spectral optimization problems"

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preprint available on `http://cvgmt.sns.it`

The problem we want to deal with is the evolution of domains according to some kinds of gradient flows which take into account functionals occurring in spectral optimization problems.

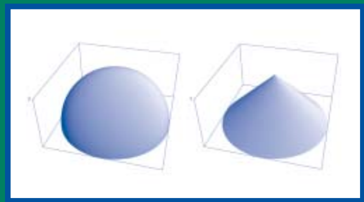
For a detailed survey on shape optimization problems of spectral type we refer to:

G. Buttazzo REMC (2011)
D. Bucur, G. Buttazzo Birkhäuser (2005)
A. Henrot Birkhäuser (2006)
A. Henrot, M. Pierre Springer (2005)

Progress in Nonlinear Differential Equations
and Their Applications

Dorin Bucur
Giuseppe Buttazzo

Variational Methods in Shape Optimization Problems



Birkhäuser

Shape optimization problems are generally written as

$$\min\Big\{F(\Omega)\ :\ \Omega\in\mathcal{A}\Big\}$$

where $\mathcal{A}$ is a suitable family of admissible domains and F is a suitable cost function defined on $\mathcal{A}$.

In particular we consider problems arising in spectral optimization: the admissible class $\mathcal{A}$ is made of domains of $\mathbf{R}^d$ and the cost functional F is of one of the following types.

Integral functionals. Denoting by u_Ω the solution (extended by zero outside of Ω) of

$$-\Delta u = f \text{ in } \Omega, \qquad u \in H^1_0(\Omega)$$

consider, for a suitable integrand j

$$F(\Omega) = \int_{\mathbf{R}^d} j\Big(x, u_\Omega(x)\Big)\, dx.$$

Spectral functionals. Denoting by $\lambda(\Omega)$ the spectrum of the Dirichlet Laplacian in Ω, consider, for a suitable function Φ.

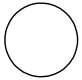
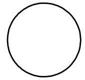
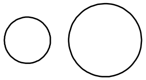
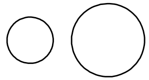
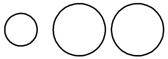
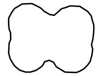
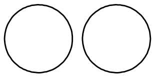
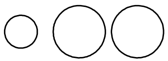
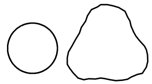
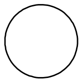
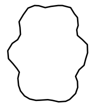
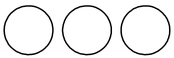
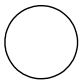
$$\min\Big\{\Phi\Big(\lambda(\Omega)\Big) \ : \ \Omega \in \mathcal{A}\Big\}.$$

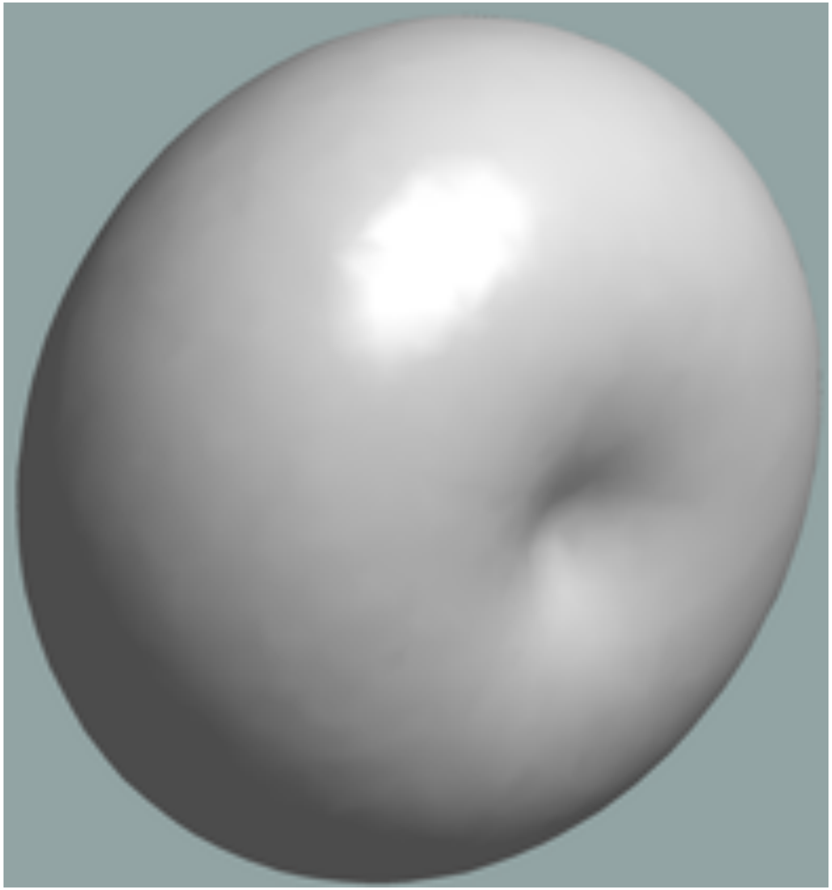
For instance, for a fixed integer k,

$$\min\Big\{\lambda_k(\Omega) \ : \ \Omega \in \mathcal{A}\Big\}.$$

The existence of minimizers for this kind of problems, as well as the related necessary conditions of optimality, have been studied a lot, see References above. Several problems still remain open, even for the existence of optimal shapes.

Here are some numerical computations (made by E. Oudet) for the optimal domains of λ_k, with prescribed measure.

No	Optimal union of discs		Computed shapes	
3		46.125		46.125
4		64.293		64.293
5		82.462		**78.47**
6		92.250		**88.96**
7		110.42		**107.47**
8		127.88		**119.9**
9		138.37		**133.52**
10		154.62		**143.45**



the best domain for λ_3 in $\mathbf{R}^3$

We want to study the shape evolution $\Omega(t)$, starting from a given domain Ω_0 according to a suitable definition of gradient flow.

The theory of minimizing movements was conceived by De Giorgi to provide a formulation of gradient flow evolutions for very general functionals defined on metric spaces.

Recent developments have been made by Ambrosio, Gigli, Savaré (Birkhäuser 2005) and can be adapted to our purposes as soon as the class of admissible domains can be endowed by a suitable dissipation distance.

The difficulty consists in the fact that a too "strong" distance does not have enough compactness and domains may evolve into relaxed shapes, while a "weak" distance makes difficult the proof of continuity (or lower semicontinuity) properties, which are crucial in the existence proofs.

Several distances on the class of domains can be considered (Hausdorff, Lebesgue, . . .) but an appropriate choice has to take into account the functional under consideration.

Let (X,d) be a metric space, $u_0 \in X$ an initial condition, and $F : X \to]-\infty, +\infty]$ a functional. For every time step $\varepsilon > 0$ the *implicit Euler scheme* of initial condition u_0 consists in constructing a function $u_\varepsilon(t) = w([t/\varepsilon])$ ($[\cdot]$ =integer part), by $w(0) = u_0$ and

$$w(n+1) \in \operatorname{argmin}\left\{F(v) + \frac{d^2(v, w(n))}{2\varepsilon}\right\}.$$

Definition We say that $u : [0,T] \to X$ is a generalized minimizing movement for F with initial condition u_0, and we write $u \in GMM(F,d,u_0)$, if for a subsequence $\varepsilon_n \to 0$

$$u_{\varepsilon_n}(t) \to u(t) \qquad \forall t \in [0,T].$$

Many applications of the minimizing movements theory have been done for quasi-static evolutions (in fracture mechanics and for evolutions in presence of friction terms) and in Hilbertian settings, where the evolution takes the form, when F is smooth,

$$u' = -\nabla F(u), \quad u(0) = u_0.$$

In general we have:

Theorem If F is d-l.s.c. with sublevels d-compact in X, then for every initial condition $u_0 \in \mathrm{dom}\,\mathrm{F}$ there exists a $GMM(F, d, u_0)$ evolution $u(t)$. Moreover $u \in AC^2(0, T; X)$, i.e. $|u'| \in L^2(0, T; X)$.

The natural framework for our shape problems is the one of quasi-open domains, i.e. domains $\Omega = \{u > 0\}$ with $u \in H^1(\mathbf{R}^d)$. For simplicity we work in a given bounded set D by considering the class

$$\mathcal{A}(D) = \{\Omega \text{ quasi-open}, \ \Omega \subset D\}.$$

An interesting distance on $\mathcal{A}(D)$ is given by

$$d_\gamma(\Omega_1, \Omega_2) = \|w_{\Omega_1} - w_{\Omega_2}\|_{L^2}$$

being w_Ω the solution of

$$-\Delta u = 1 \text{ in } \Omega, \qquad u \in H^1_0(\Omega).$$

Let $\mathcal{R}_\Omega$ be the resolvent operator of $-\Delta$ in $H^1_0(\Omega)$, so that $w_\Omega = \mathcal{R}_\Omega(1)$.

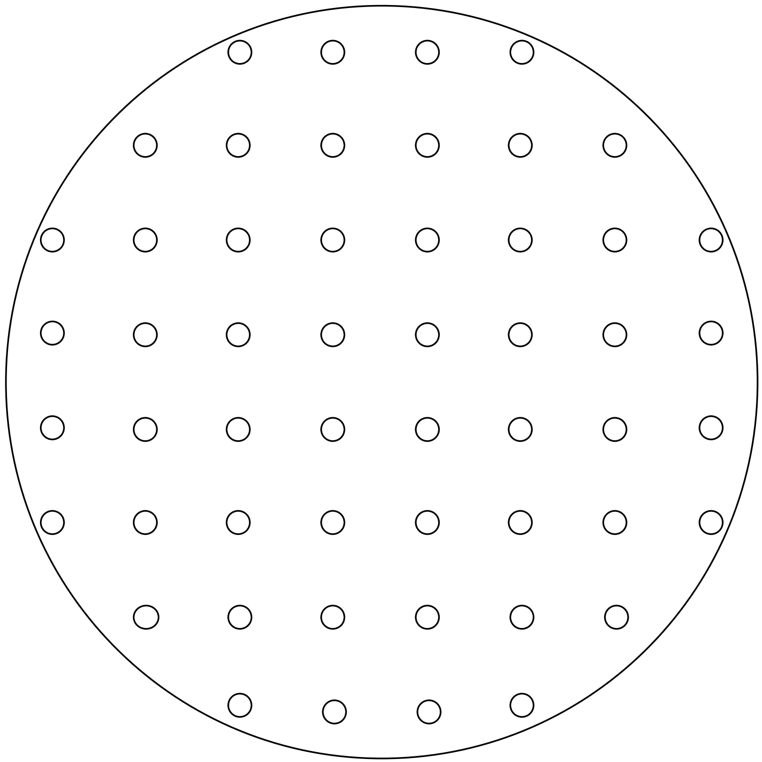
properties of γ-convergence

• $\Omega_n \to_\gamma \Omega$ iff $\mathcal{R}_{\Omega_n} \to \mathcal{R}_\Omega$ in the $\mathcal{L}\Big(L^2(D)\Big)$ operator norm. Then many integral functionals are γ-continuous or γ-l.s.c.

• The γ-convergence is very strong; for instance the spectrum $\lambda(\Omega)$ is γ-continuous.

• The γ-convergence is metrizable on $\mathcal{A}(D)$ but it is not compact; its compactification coincides with the class of capacitary measures

$$\mathcal{M}_0(D) = \{\mu(B) = 0 \text{ whenever } \operatorname{cap}(B) = 0\}.$$



The sets Ω_n in the Cioranescu-Murat example; their γ-limit is the Lebesgue measure.

By considering the resolvent operators $\mathcal{R}_\mu(f)$ of the PDE

$$-\Delta u + \mu u = f, \qquad u \in H_0^1(D) \cap L^2_\mu$$

the space $\mathcal{M}_0(D)$ can be endowed by the γ-convergence:

$$d_\gamma(\mu_1, \mu_2) = \|\mathcal{R}_{\mu_1}(1) - \mathcal{R}_{\mu_2}(1)\|_{L^2}.$$

- $\mathcal{M}_0(D)$ is a compact metric space;
- smooth domains of $\mathcal{A}(D)$ are γ-dense;
- measures $a(x)\,dx$, a smooth, are γ-dense;
- the spectrum of $-\Delta + \mu$ is γ-continuous.

Theorem *Let* $F : \mathcal{M}_0(D) \to]-\infty, +\infty]$ *be* γ-*l.s.c. and* $\mu_0 \in \mathrm{dom}\,\mathsf{F}$*. Then there exists* $\mu \in GMM(F, \gamma, \mu_0)$ *and* $\mu \in AC^2(0, T; \mathcal{M}_0(D))$*.*

There is a natural one-to-one map between $\mathcal{M}_0(D)$ and the convex set

$$K = \{w \in H_0^1(D) \ : \ w \geq 0, \ 1 + \Delta w \geq 0\} \subset L^2(D),$$

given by

$$\mu \mapsto w_\mu = \mathcal{R}_\mu(1) \text{ with inverse } w \mapsto \mu_w = \frac{1 + \Delta w}{w}.$$

Moreover, the metric structure on $\mathcal{M}_0(D)$ and K is the same, since

$$d_\gamma(\mu_1, \mu_2) = \|w_{\mu_1} - w_{\mu_2}\|_{L^2(D)}.$$

Therefore, every functional $F : \mathcal{M}_0(D) \to]-\infty, +\infty]$ can be identified with a functional $J : L^2(D) \to]-\infty, +\infty]$ with $\mathrm{dom}\, \mathsf{J} \subset \mathsf{K}$

$F(\mu) = J(w_\mu)$ or equivalently $J(w) = F(\mu_w)$.

The variational flow for F in $\mathcal{M}_0(D)$ can be then obtained through the gradient flow of J in $L^2(D)$, generated by the implicit Euler scheme

$$w_\varepsilon^{n+1} \in \mathrm{argmin}\left\{ J(w) + \frac{1}{2\varepsilon}\int_D |w - w_\varepsilon^n|^2\, dx \right\}.$$

Question: if we consider the GMM associated to the energy functional $J(w) = -\int_D w(x)\,dx$ and start from a domain Ω_0, will the flow remain in the family of domains or do the evolution occur in the space of measures?

In general a relaxation phenomenon occurs, at least at the discrete level. This kind of phenomenon was numerically observed also in the framework of quasi-static debonding membranes [Bucur-Buttazzo-Lux, ARMA 2008], where the evolution takes place in the family of *relaxed domains* (mushy regions).

If we want to "force" the evolution to take place in the family of domains, either we have to add geometrical constraints on the space of shapes (as convexity, equi-Lipschitz property, uniform exterior cone condition, . . .) or we have to modify the dissipation distance.

Another natural distance is given by the Lebesgue measure of the symmetric difference

$$d_{char}(\Omega_1, \Omega_2) = |\Omega_1 \triangle \Omega_2|.$$

Since two quasi-open sets may differ for a negligible set (think for instance in $\mathbf{R}^2$ to a disk and a disk minus a segment), this is not a proper metric in $\mathcal{A}(D)$, so that one should consider equivalence classes in the family of shapes.

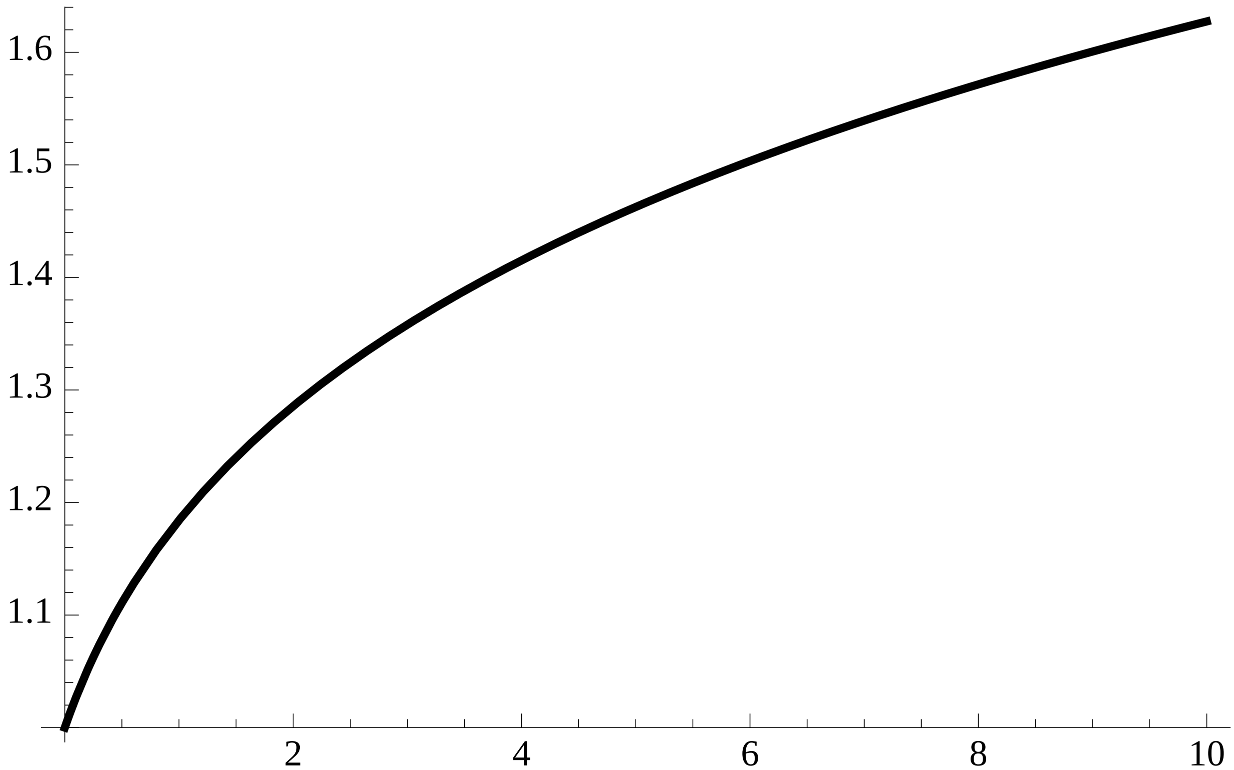
Theorem *Let* $F : \mathcal{A}(D) \to]-\infty, +\infty]$ *be* γ*-l.s.c and monotone decreasing for set inclusion. Then, for every* $\Omega_0 \in \mathrm{dom}\,\mathsf{F}$ *there exists a* $GMM(F, d_{char}, \Omega_0)$ *evolution* $\Omega(t)$*. Moreover,* $\Omega(t)$ *is increasing in the sense of set inclusion.*

Evolution of a ball Let $\Omega_0 = B(0, R_0)$ and take $F(\Omega) = \lambda_1(\Omega)$. We may prove that for every t the set $\Omega(t)$ is still a ball centered at the origin, of radius $R(t)$, with

$$R(t) = \left(R_0^{2d+2} + \frac{4(d+1)\lambda_1(B(0,1))}{d^2\omega_d^2} t\right)^{1/(2d+2)}$$

For instance in the two-dimensional case, starting from the unit ball $R_0 = 1$, we have

$$R(t) = (1 + 1.76\, t)^{1/6}.$$



Evolution of the radius of a ball for the functional $\lambda_1(\Omega)$ starting from $R_0 = 1$.

Some properties of the flow:

- The map $t \mapsto F(\Omega(t))$ may be discontinuous.

- Topological genus is not preserved.

- Balls evolve into balls (no symmetry breaking) but convexity is not preserved (convexity breaking).

Some open questions:

- If Ω_0 is convex, under which conditions on Ω_0 and m the minimizers of

$$\min\left\{\lambda_1(\Omega)\ :\ \Omega_0\subset\Omega,\ |\Omega|=m\right\}$$

are still convex? According to the argument above, if Ω_0 is a square and m is slightly larger than $|\Omega_0|$, then the optimal domains should not be convex.

- Is it true that the evolution associated to λ_1 converges (after rescaling) as $t\to+\infty$ to a ball?

- More generally, is it true that the evolution associated to λ_k converges (after rescaling) as $t \to +\infty$ to the minimizer of λ_k?

- Does the convergence above (if any) occur in a finite time?

- Is it true that for a spectral shape functional like $\lambda_k(\Omega)$ every stationary shape is a minimizer?

- Prove or disprove that the metric derivative of λ_1 computed at a bounded smooth set Ω is given by

$$|\lambda_1'|(\Omega) = \max_{\partial\Omega} \left|\frac{\partial u_1}{\partial n}\right|^2.$$

Precisely, prove that

$$\limsup_{|\Omega_n \setminus \Omega| \to 0,\ \Omega \subset \Omega_n} \frac{\lambda_1(\Omega) - \lambda_1(\Omega_n)}{|\Omega_n \setminus \Omega|} \leq \max_{\partial\Omega} \left|\frac{\partial u_1}{\partial n}\right|^2.$$

Variations on the theme

• Constraint on the measure. An alternative evolution is to work in the class $\{|\Omega| = m\}$. The incremental problem is then

$$\Omega_\varepsilon^{n+1} \in \text{argmin}_{\Omega\in\mathcal{A}(D),|\Omega|=m} \left\{ F(\Omega) + \frac{1}{2\varepsilon} |\Omega_\varepsilon^n \triangle \Omega|^2 \right\}$$

The existence of a flow $\Omega(t)$ is not clear. Nevertheless, one can construct discrete solutions of the incremental scheme.
An alternative way is to replace the measure constraint by adding a penalization term in the functional, i.e. to replace $F(\Omega)$ by $F(\Omega) + c|\Omega|$.

• Perimeter penalization. One can alternatively introduce a penalization on the perimeter. In this case, the incremental step reads

$$\Omega_\varepsilon^{n+1} \in \mathrm{argmin}_{\Omega\in\mathcal{A}(D)} \left\{ F(\Omega) + P_D(\Omega) + \frac{1}{2\varepsilon} |\Omega_\varepsilon^n \triangle \Omega|^2 \right\}$$

where $P_D(\Omega)$ denoted the perimeter of Ω into D.

The topology given by d_{char} turns out to be compact on the sublevels of $F + P_D$.

• Hausdorff complementary distance. If we consider the Hausdorff complementary distance

$$d_{H^c}(\Omega_1, \Omega_2) = \max_{x \in \overline{D}} |d(x, \overline{D} \backslash \Omega_1) - d(x, \overline{D} \backslash \Omega_2)|,$$

and F is increasing for the set inclusion, then, there exists a flow of the form

$$\Omega(t) = D \setminus (\Omega_0^c + \overline{B}_{f(t)}),$$

with f is continuous and increasing.

• Hausdorff distance. Similarly, for the Hausdorff distance

$$d_H(F_1, F_2) = \max_{x \in \overline{D}} |d(x, F_1) - d(x, F_2)|$$

and functionals F decreasing for the set inclusion, the existence of a flow of the form

$$\Omega(t) = \mathrm{int}(\overline{\Omega}_0 + \overline{B}_{f(t)})$$

can be obtained, with f continuous and increasing.

• Flows of convex shapes. We may restrict the evolution to convex domains:

$$\mathcal{K}(D) = \{K \subset D \ : \ K \text{ open and convex}\}.$$

There are various distances on K, all topologically equivalent:

- the Hausdorff distance;

- $d_2(K_1, K_2) = \|b_{K_1} - b_{K_2}\|_{L^2(D)}$, where b_K is the oriented distance function;

- the L^1 distance of the characteristic functions $d_{char}(K_1, K_2) = |K_1 \triangle K_2|$.

Let $F : \mathcal{K}(D) \to \mathbf{R}$ be a γ-lower semicontinuous shape functional which satisfies the coercivity assumption:

$$F(K_n) \to +\infty \quad \text{whenever } |K_n| \to 0.$$

Notice that all previous topologies are compact on sublevels of F. Therefore we have:

Theorem For each of the distances d_H, d_{char}, d_2, and for every initial convex set $K_0 \in D(F)$, there exists an evolution flow $K(t)$ starting from K_0.