On a notion of solution to a forward-backward parabolic equation in one space dimension, and its regularizations

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Collaboration with

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[BBMN]: *Convergence of the one-dimensional Cahn-Hilliard equation*, SIAM J. Math. Anal., to appear.

- M. Novaga (Univ. Padova, Italy)
- C. Geldhauser (Univ. Bonn, Germany)

[BGN]: *Convergence of a semidiscrete scheme for a forward-backward parabolic equation*, forthcoming.

Plan of the talk

- motivations
- the ill-posed parabolic problem.
- the viscous regularization.
- the semidiscrete regularization.
- passing to the limit for a class of initial data precisely leading to formation of microstructures.
- a notion of solution of a new equation.

MOTIVATIONS

• Anisotropic mean curvature flow, i.e. (formal) gradient flow of

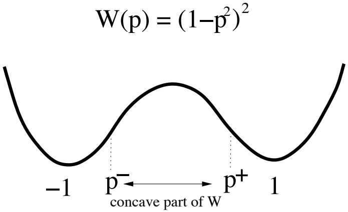
$$\partial E \to \int_{\partial E} \gamma(\nu)$$

when the anisotropy γ (Frank diagram $= \{\gamma = 1\}$) is nonconvex. Numerical observations [M. Paolini et al.] show quick formation of microstructures (instabilities) in the **local unstable region** of ∂E, i.e., regions of ∂E where ν (normalized) belongs to the **concave** part of the Frank diagram, and *not* in the whole region where ν belongs to $\{\gamma = 1\} \neq \{\gamma^{**} = 1\}$.

• *The evolution is not, in general, the evolution obtained by convexifying* γ.

• Ill-posed PDEs: forward-backward parabolic equations. Regularizations may lead (in the limit) to a free boundary problem.

THE PROBLEM



The formal gradient flow of $F(u) = \int_{\mathbb{T}} W(u_x)\, dx$

$$\begin{cases} u_t = (W'(u_x))_x & \text{in } \mathbb{T} \times [0, T] \\ u(0) = \overline{u}. & \text{on } \mathbb{T} \times \{0\} \end{cases}$$

ill-posed. Typical source of instabilities when

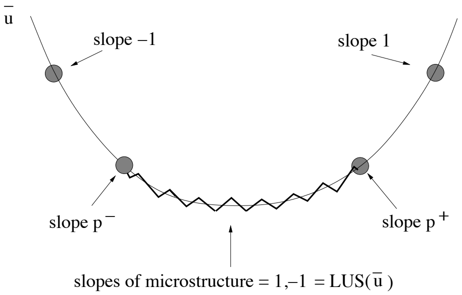
$$\overline{u}_x(x) \in (p^-, p^+) := \{W'' < 0\} \text{ **concave part** of } W$$

Problem: can we define a reasonable notion of solution in these cases?

Define the **local unstable region** of $\overline{u}$,

$$\mathrm{LUS}(\overline{u}) := \Big\{ x \in \mathbb{T} : \overline{u}_x(x) \in \{W'' < 0\} \Big\}$$

Our problem is: can we flow $\overline{u}$ with $\mathrm{LUS}(\overline{u}) \neq \emptyset$, initial data precisely leading to microstructures?



Natural idea: regularize the equation. Different regularizations could lead to different solutions.

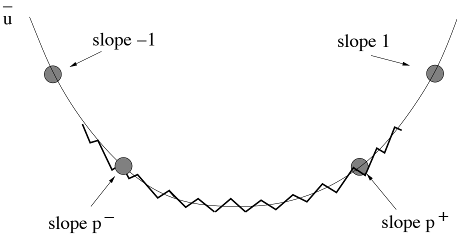
- [Plotnikov], [Barenblatt et al.], [Evans & Portilheiro], [Smarrazzo & Tesei]

$$u_t = \epsilon^2 u_{txx} + (W'(u_x))_x$$

or, with $v = u_x$, $v_t = \epsilon^2 v_{txx} + (W'(v))_{xx}$.

- [Slemrod], [De Giorgi], [B., Fusco & Guglielmi]: viscous regularization

$$u_t = -\epsilon^2 u_{xxxx} + (W'(u_x))_x, \qquad u(0) = \overline{u}_\epsilon$$



Artificially imposing a microstructure in a region $\supset \mathrm{LUS}(\overline{u})$ seems to change the solution.

Pass to the Cahn-Hilliard equation with $v := u_x$,

$$v_t = -\left(\epsilon^2 v_{xx} + W'(v)\right)_{xx}, \qquad v(0) = \overline{v}_\epsilon$$

Theorem (BBMN)

$\exists \lim_{\epsilon \to 0} v_\epsilon = v$ *solution of the convexified problem*

$$v_t = (W^{**\prime}(v))_{xx}$$

if we energetically prepare $\overline{v}$.

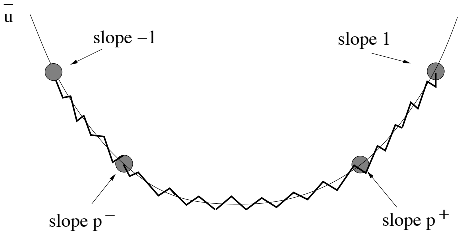
Set

$$F_\epsilon(v) := \int_{\mathbb{T}} \left[\epsilon (v_x)^2 + \frac{1}{\epsilon} W(v)\right]$$

Energetically preparing $\overline{v}$ means

$$F_\epsilon(\overline{v}_\epsilon) \to F^{**}(v) := \int_{\mathbb{T}} W^{**}(v).$$

Roughly, this means to superimpose the microstructure on the whole region between slopes -1 and 1.



Proof: we follow an idea of [Sandier & Serfaty] based on coerciveness and Γ-liminf property of the functionals

$$\int_{\mathbb{T}} \left[\Big(W'(v) - \epsilon^2 v_{xx} \Big)_x \right]^2$$

• Ttheorem [BBCN] does not solve our original problem. What happens if we superimpose the microstructure in an intermediate region containing $\mathrm{LUS}(\overline{u})$?

- Semidiscrete discretization:

$$\begin{cases} \dfrac{du^h}{dt} = D_h^+ \ W'(D_h^- u^h) & \text{in } \mathbb{T} \times [0, +\infty) \\ \\ u^h(0) = \overline{u}^h & \text{on } \mathbb{T} \times \{0\} \end{cases}$$

$h > 0$ grid size, $D_h^\pm$ difference quotients, $\overline{u}^h$ discrete initial data converging to $\overline{u}$ as $h \to 0^+$.

• a compactness property cannot be obtained for $D_h^- u^h$: u^h have oscillations which are typically of order h in $\mathrm{LUS}(u^h)$.

Theorem (Geldhauser & Novaga (2011))

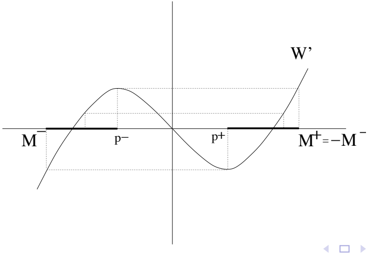
• $\exists \lim_{h\to 0^+} u^h$ *Lipschitz function* u, *when the gradients of* $\overline{u}$ *and* $\overline{u}^h$ *lie in* $\{W'' \geq 0\}$, *with possible jumps from one connected component to another* ($\mathrm{LUS}(\overline{u})$ *is pointwise) with continuity of* $W'(u_x)$.

• *Preserving of the* M^+ *Lipschitz constant: the condition*

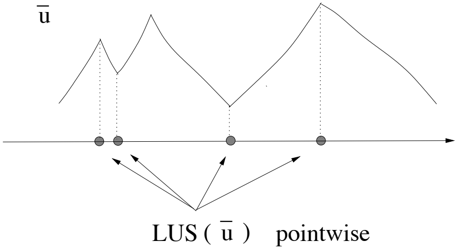
$$|\overline{u}_x(x)| \leq M^+ \qquad \text{is preserved}$$

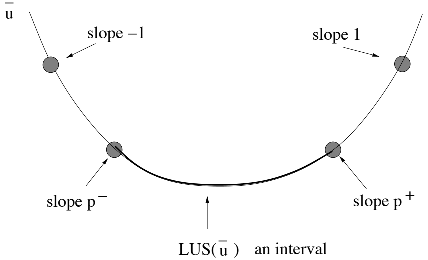
• *Avoidance preserving property of* LUS:

$$\mathrm{LUS}(\overline{u}) = \emptyset \quad \Rightarrow \quad \mathrm{LUS}(u(t)) = \emptyset$$



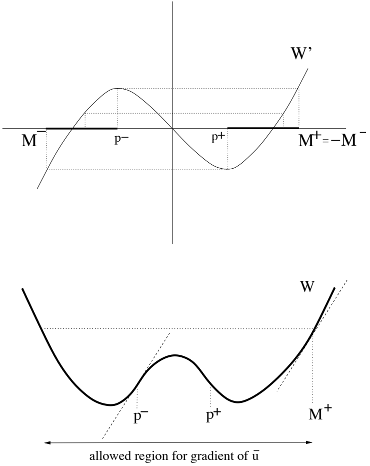
Theorem [NG] does not solve our problem. We need to remove the assumption that the gradient of $\overline{u}$ belongs to $\{W'' \geq 0\}$.





From now on we suppose

$$|\overline{u}_x(x)| \leq M^+, \qquad x \in \mathbb{T}$$



We will pass to the limit and characterize the limit equation. Approximate $\overline{u}$ with any $\overline{u}^h$ so that

$$\mathrm{LUS}(\overline{u}^h) = \emptyset$$

This approximation can be made in several different ways. The limit solution could depend on the approximation we choose: this will be the case.

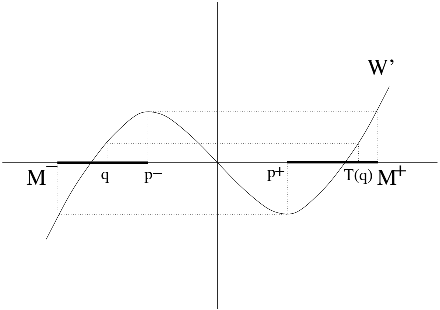
- How to find conditions on $\overline{u}$ so that $\overline{u}^h$ have $\mathrm{LUS}(\overline{u}^h) = \emptyset$?
  Endow $\overline{u}$ with an auxiliary function

$$\overline{\varrho} \in L^\infty(\mathbb{T}; [0,1])$$

  percentage of mesh intervals where $\overline{u}^h$ is decreasing in a neighborhood of the point $x \in \mathbb{T}$.
- the (discrete) gradients of $\overline{u}^h$ must be a combination of allowed gradients; this is dictated by $\overline{\varrho}$.
- use the avoidance preserving properties of LUS to ensure that $\mathrm{LUS}(u^h(t)) = \emptyset$.
- find a new scale, larger than h, so that the gradients of $u^h(t)$, averaged at this scale, converge. The scale is dictated by $\overline{\varrho}$.
- pass to the limit. The new scale is source of a new limit equation, of the form

$$u_t = \Big(W'(q(u_x, \overline{\varrho}))\Big)_x$$

Aim: for fixed $p \in [M^-, M^+]$ and $\sigma \in [0,1]$, define $q(p,\sigma)$ appearing in the new equation ($p = u_x(x,t)$, $\sigma = \overline{\varrho}(x)$).



First define the map T

$$\mathrm{T} : [M^-, M^+] \setminus (p^-, p^+) \to [M^-, M^+] \setminus (p^-, p^+)$$

given q, $\mathrm{T}(q)$ is such that

$$W'(q) = W'(\mathrm{T}(q))$$

Given $p \in [-M^-, M^+]$ and $\sigma \in [0,1]$ solve in q the problem:

$$q \in [M^-, M^+] \setminus (p^-, p^+)$$

$$\sigma q + (1-\sigma)\mathrm{T}(q) = p$$

Define $q = q(p, \sigma)$ as the smallest between q and $\mathrm{T}(q)$, which is negative.

Definition

Given $\overline{u}$, a percentage $\overline{\varrho}$ of negative slopes is any function in $L^\infty(\mathbb{T}; [0,1])$ such that

$$q(\overline{u}_x(x), \overline{\varrho}(x)) < 0$$

$$\overline{\varrho}(x) q(\overline{u}_x(x), \overline{\varrho}(x)) + (1 - \overline{\varrho}(x))\mathrm{T}(q(\overline{u}_x(x), \overline{\varrho}(x))) = \overline{u}_x(x)$$

• A percentage $\overline{\varrho}$ of negative slopes always exists.

• Not uniquely determined. Since limit solution u depends on $\overline{\varrho}$, u not uniquely determined. Also when $\mathrm{LUS}(\overline{u}) = \emptyset$.

• $\mathrm{LUS}(\overline{u}) = \emptyset$

$$\overline{\varrho} \equiv 0 \quad \text{or} \quad \overline{\varrho} \equiv 1$$

• $\mathrm{LUS}(\overline{u})$ pointwise

$$\overline{\varrho}(x) \in \{0, 1\}$$

• $\overline{u}$ with alternate regions where $\overline{u}_x$ is in $\{x < 0 : W''(x) > 0\}$, in $\{W'' < 0\}$ and in $\{x > 0 : W''(x) > 0\}$

$$\overline{\varrho}(x) \in \left\{0, \frac{1}{2}, 1\right\}$$

$1/2$ in $\mathrm{LUS}(\overline{u})$.

Construction of $\overline{u}^h$ from $\overline{\varrho}$. Assume

$$\overline{\varrho}(x) = \sigma = \frac{m}{n} \in [0,1], \qquad x \in I$$

We partition I into subintervals of length nh: in $[knh, (k+1)nh]$, the averaged slope of $\overline{u}$ is given by

$$(D^-_{nh}\, u^h)_i := \frac{u^h_i - u^h_{i-n}}{nh}$$

We impose:

- $\overline{u}^h$ has the same averaged slope of $\overline{u}$ in $[knh, (k+1)nh]$;
- $\mathrm{LUS}(\overline{u}^h) = \emptyset$;
- $\overline{u}^h$ is decreasing with slope $\overline{q}_k$ on m mesh subintervals of $[knh, (k+1)nh]$, and increasing with slope $\mathrm{T}(\overline{q}_k)$ on the remaining $n - m$ mesh subintervals, where

$$\overline{q}_k := q\,(\overline{p}_k, \sigma) < 0, \qquad p_k := (D^-_{nh}\, \overline{u})_{(k+1)n}$$

Theorem (BGN)

Let $\overline{u}$ and $\overline{\varrho}$ be given. Let $\overline{u}^h$ be as above. Then the averaged discrete gradients of u^h are compact, and $u^h \to u$, where u solves distributionally

$$\begin{cases} u_t = \Big(W'(q(u_x, \overline{\varrho}))\Big)_x \\ u(0) = \overline{u} \end{cases} \tag{1}$$

- This result covers the case of a specific choice of $\overline{\varrho}$, and gives a notion of solution to the original ill-posed problem, for a large class of initial data $\overline{u}$.
- No uniqueness for solutions obtained as limits of the semidiscrete scheme approximating $\overline{u}$ with $\overline{u}^h$ so that $\mathrm{LUS}(\overline{u}^h) = \emptyset$.
- There seems not to be a choice of σ reproducing the solution numerically observed with the viscous regularization of fourth order.