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# EXPLICIT SOLUTIONS FOR STEFAN-LIKE PROBLEMS WITH CONVECTIVE BOUNDARY CONDITION

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## ABSTRACT

We give the conditions on data in order to have explicit solutions for Stefan-like problems with a particular convective boundary condition at the fixed face for a semi-infinite material in the following cases:

1) Classical two-phase solidification problem; Equivalence between convective and temperature boundary conditions at the fixed face. Inequality for the coefficient which characterizes the free boundary for the Neumann solution;

2) Classical two-phase solidification problem with density jump; Quilghini transformation to reduce the problem “with density jump” into a problem “without density jump”;

3) One-phase melting problem with the Solomon-Wilson-Alexiades model of mushy region; Determination of one unknown thermal coefficient with an over-specified condition at the fixed face;

4) Two-phase solidification problem with a mushy zone model;

5) Rubinstein binary-alloy solidification problem;

6) Thawing in a saturated porous medium by considering a density jump and the influence of the pressure on the melting temperature (in progress).

These explicit solutions complement the ones given recently in D.A. Tarzia, “Explicit and Approximated Solutions for Heat and Mass Transfer Problems with a Moving Interface”, In Advanced Topics in Mass Transfer, Mohamed El-Amin (Ed.), InTech Open Access Publisher, Rijeka (2011), Chapter 20, pp. 439-484. Available from:

http://www.intechopen.com/articles/show/title/explicit-and-approximated-solutions-for-heat-and-mass-transfer-problems-with-a-moving-interface

**MOTIVATION:**

S.M. Zubair – M.A. Chaudhry, Wärme und Stoffübertragung (now Heat and Mass Transfer), 30 (1994), 77-81.

**GOAL:**

The goal of this work is to find the necessary and/or sufficient conditions for data (initial and boundary conditions, and thermal coefficients) in order to obtain an instantaneous phase-change with the corresponding explicit solution of the similarity type when a convective boundary condition is impossed on the fixed face.

## 1) CLASSICAL TWO-PHASE SOLIDIFICATION PROBLEM

We consider the following free boundary problem: find the solid-liquid interface $x = s(t)$ and the temperature $T(x,t)$ defined by

$$T(x,t) = \begin{cases} T_s(x,t) & \text{if } \ 0 < x < s(t), \quad t > 0, \\ T_f & \text{if } \ x = s(t), \quad t > 0, \\ T_\ell(x,t) & \text{if } \ x > s(t), \quad t > 0, \end{cases} \tag{1}$$

which satisfy the following equations and boundary conditions

$$T_{s_t} = \alpha_s T_{s_{xx}}, \quad 0 < x < s(t), \ t > 0 \tag{2}$$

$$T_{\ell_t} = \alpha_\ell T_{\ell_{xx}}, \quad x > s(t), \ t > 0 \tag{3}$$

$$T_s(s(t),t) = T_\ell(s(t),t) = T_f, \quad x = s(t), \quad t > 0 \tag{4}$$

$$k_s T_{s_x}(s(t),t) - k_\ell T_{\ell_x}(s(t),t) = \rho \ell \dot{s}(t), \quad x = s(t), \quad t > 0 \tag{5}$$

$$T_\ell(x,0) = T_\ell(+\infty,t) = T_i, \quad x > 0, \quad t > 0 \tag{6}$$

$$s(0) = 0 \tag{7}$$

$$k_s T_{s_x}(0,t) = \frac{h_0}{\sqrt{t}} (T_s(0,t) - T_\infty), \quad t > 0 \tag{8}$$

where the subscripts $s$ and $\ell$ represent the solid and liquid phases respectively, $\rho$ is the common density of mass, $k$ is the thermal conductivity, $\alpha = k / \rho c$ is the thermal diffusivity, and $\ell$ is the latent heat of fusion, and $T_\infty < T_f < T_i$. We have the following results:

**Theorem 1** (T, MAT-Serie A (2004))

If the coefficient $h_0$ verifies the inequality

$$h_0 > \frac{k_\ell}{\sqrt{\pi \alpha_\ell}} \frac{T_i - T_f}{T_f - T_\infty} \tag{9}$$

there exists an instantaneous solidification process and then the free boundary problem (2)-(8) has the explicit solution to a similarity type given by

$$s(t) = 2\lambda \sqrt{\alpha_\ell t} \tag{10}$$

$$T_s(x,t) = T_\infty + \frac{(T_f - T_\infty)\left[1 + \frac{h_0 \sqrt{\pi \alpha_s}}{k_s} erf\left(\frac{x}{2\sqrt{\alpha_s t}}\right)\right]}{1 + \frac{h_0 \sqrt{\pi \alpha_s}}{k_s} erf\left(\lambda \sqrt{\frac{\alpha_\ell}{\alpha_s}}\right)} \tag{11}$$

$$T_\ell(x,t) = T_i - (T_i - T_f)\frac{erfc\left(\frac{x}{2\sqrt{\alpha_\ell t}}\right)}{erfc(\lambda)} \tag{12}$$

and the dimensionless parameter $\lambda > 0$ satisfies the following equation:

$$F(x) = x, \quad x > 0 \tag{13}$$

where function $F$ and the $b$'s coefficients are given by

$$F(x) = b_1 \frac{\exp(-bx^2)}{1 + b_2 erf(x\sqrt{b})} - b_3 \frac{\exp(-x^2)}{erfc(x)} \tag{14}$$

$$b = \frac{\alpha_\ell}{\alpha_s} > 0; \quad b_1 = \frac{h_0(T_f - T_\infty)}{\rho\ell\sqrt{\alpha_\ell}} > 0 \tag{15}$$

$$b_2 = \frac{h_0}{k_s}\sqrt{\pi\alpha_s} > 0; \quad b_3 = \frac{c_\ell(T_i - T_f)}{\ell\sqrt{\pi}} > 0. \tag{16}$$

**EQUIVALENCE BETWEEN CONVECTIVE AND TEMPERATURE BOUNDARY CONDITIONS AT THE FIXED FACE. THE INEQUALITY FOR THE COEFFICIENT WHICH CHARACTERIZES THE FREE BOUNDARY FOR THE NEUMANN SOLUTION**

When we consider the following temperature boundary condition:

$$T_s(0,t) = T_0, \quad t > 0 \quad (T_0 < T_f < T_i) \tag{17}$$

instead of the convective boundary condition (8), the free boundary problem (2)-(7) and (17) has the classical Neumann solution whose solid-liquid interface is given by the expression:

$$s(t) = 2\xi\sqrt{\alpha_\ell t}, \quad t > 0, \tag{18}$$

where the coefficient $\xi > 0$ is the unique solution of an adequate equation.

If we define the relation between $T_0 > 0$, and $T_\infty > 0$ and $\mathrm{h}_0 > 0$ by the following expression:

$$T_0 = \frac{T_f + T_\infty \dfrac{h_0\sqrt{\pi\alpha_s}}{k_s} erf(\lambda\sqrt{\dfrac{\alpha_\ell}{\alpha_s}})}{1 + \dfrac{h_0\sqrt{\pi\alpha_s}}{k_s} erf(\lambda\sqrt{\dfrac{\alpha_\ell}{\alpha_s}})} \qquad (19)$$

then we obtain the following result:

**Theorem 2**

Taking into account the relation (19), the free boundary problems (2)-(8), and (2)-(7) and (17) are equivalent (i.e. $\lambda = \xi$, and the solid and liquid temperatures and the free boundaries are coincident). Moreover, the coefficient $\xi > 0$, which characterizes the free boundary of the Neumann solution, satisfies the following inequality:

$$erf\left(\xi\sqrt{\frac{\alpha_\ell}{\alpha_s}}\right) < \frac{k_s}{k_\ell}\sqrt{\frac{\alpha_\ell}{\alpha_s}}\,\frac{T_f - T_\infty}{T_0 - T_\infty}\,\frac{T_f - T_0}{T_i - T_f}, \quad \forall T_0 \in \left(T_\infty, T_f\right). \qquad (20)$$

## 2) CLASSICAL TWO-PHASE SOLIDIFICATION PROBLEM WITH DENSITY JUMP

We consider the following free boundary problem: find the solid-liquid interface $x = s(t)$ and the temperature $T(x,t)$ defined by

$$T(x,t)=\begin{cases} T_s(x,t) & \text{if} \quad 0<x<s(t), \quad t>0, \\ T_f & \text{if} \quad x=s(t), \quad t>0, \\ T_\ell(x,t) & \text{if} \quad x>s(t), \quad t>0, \end{cases} \tag{21}$$

which satisfy the following equations and boundary conditions

$$T_{s_t} = \alpha_s T_{s_{xx}}, \quad 0<x<s(t), \ t>0 \tag{22}$$

$$T_{\ell_t} = \alpha_\ell T_{\ell_{xx}} + \frac{\rho_s - \rho_\ell}{\rho_\ell}\dot{s}(t) T_{\ell_x}, \quad x>s(t), \ t>0 \tag{23}$$

$$T_s(s(t),t) = T_\ell(s(t),t) = T_f, \quad x=s(t), \quad t>0 \tag{24}$$

$$k_s T_{s_x}(s(t),t) - k_\ell T_{\ell_x}(s(t),t) = \rho_s \,\ell\, \dot{s}(t), \quad x=s(t), \quad t>0 \tag{25}$$

$$T_\ell(x,0) = T_\ell(+\infty,t) = T_i, \quad x>0, \quad t>0 \tag{26}$$

$$s(0)=0 \tag{27}$$

$$k_s T_{s_x}(0,t) = \frac{h_0}{\sqrt{t}}\left(T_s(0,t) - T_\infty\right), \quad t>0 \tag{28}$$

where the subscripts $s$ and $\ell$ represent the solid and liquid phases respectively, $\rho$ is the density of mass $(\rho_s \neq \rho_\ell)$, $k$ is the thermal conductivity, $\alpha = k / \rho c$ is the thermal diffusivity, and $\ell$ is the latent heat of fusion, and $T_\infty < T_f < T_i$. We have the following results:

**Theorem 3**. If the coefficient $h_0$ verifies the inequality

$$h_0 > \frac{k_\ell}{\sqrt{\pi \alpha_\ell}} \frac{T_i - T_f}{T_f - T_\infty} \tag{29}$$

there exists an instantaneous solidification process and then the free boundary problem (22)-(28) with $\rho_s \neq \rho_\ell$ has the explicit solution to a similarity type given by

$$s(t) = 2\lambda \sqrt{\alpha_s t} \tag{30}$$

$$T_s(x,t) = T_\infty + \frac{T_f - T_\infty}{1 + \frac{h_0 \sqrt{\pi \alpha_s}}{k_s} erf(\lambda)} \left[ 1 + \frac{h_0 \sqrt{\pi \alpha_s}}{k_s} erf\left(\frac{x}{2\sqrt{\alpha_s t}}\right) \right] \tag{31}$$

$$T_\ell(x,t) = T_i - \frac{T_i - T_f}{erfc\left( \delta + \lambda \sqrt{\frac{\alpha_s}{\alpha_\ell}} \right)} erfc\left( \delta + \frac{x}{2\sqrt{\alpha_\ell t}} \right) \tag{32}$$

where

$$\varepsilon = \frac{\rho_s - \rho_\ell}{\rho_\ell}, \quad \delta = \varepsilon \xi \sqrt{\frac{\alpha_s}{\alpha_\ell}} \tag{33}$$

and the dimensionless parameter $\lambda > 0$ satisfies the following equation:

$$G(x) = x, \quad x > 0 \tag{34}$$

where function $G$ and the d'*s* coefficients are given by

$$G(x) = d_1 \frac{\exp(-x^2)}{1 + d_2 erf(x)} - d_3 \frac{\exp(-d^2 x^2)}{erfc(dx)} \tag{35}$$

$$d = \frac{\alpha_\ell}{\alpha_0} > 0; \quad d_1 = \frac{h_0 (T_f - T_\infty)}{\rho_s \ell \sqrt{\alpha_\ell}} > 0; \quad \alpha_0 = \frac{\alpha_\ell}{(1+\varepsilon)^2} \tag{36}$$

$$d_2 = \frac{h_0}{k_s} \sqrt{\pi \alpha_s} > 0; \quad d_3 = \frac{k_\ell (T_i - T_f)}{\rho_s \ell \sqrt{\pi \alpha_s \alpha_\ell}} > 0. \tag{37}$$

## QUILGHINI TRANSFORMATION TO REDUCE THE PROBLEM "WITH DENSITY JUMP" INTO A PROBLEM "WITHOUT DENSITY JUMP"

The Quilghini transformation (An. Mat. Pura Appl.(1965)) introduces the mass as a space variable by defining the following change of variables:

$$\begin{cases} e(t) = \rho_s s(t), \quad t > 0, \\ V_s(y,t) = T_s\left(\dfrac{y}{\rho_s}, t\right) \quad \text{if } 0 < x < s(t), \quad t > 0, \\ V_\ell(y,t) = T_l\left(\dfrac{\rho_\ell - \rho_s}{\rho_\ell} s(t) + \dfrac{y}{\rho_\ell}, t\right) \quad \text{if } x > s(t), \quad t > 0, \end{cases} \tag{38}$$

**Theorem 4**. (a) By the Quilghini transformation the free boundary problem (22)-(28) is given by the following one without density jump, i.e.:

$$V_{s_t} = D_s V_{s_{xx}}, \quad 0 < y < e(t), \; t > 0 \tag{39}$$

$$V_{\ell_t} = D_\ell V_{\ell_{xx}}, \quad y > e(t), \; t > 0 \tag{40}$$

$$V_s(e(t),t) = V_\ell(e(t),t) = T_f, \quad y = e(t), \quad t > 0 \tag{41}$$

$$K_s V_{s_x}(e(t),t) - K_\ell V_{\ell_x}(e(t),t) = \ell \dot{e}(t), \quad y = e(t), \quad t > 0 \tag{42}$$

$$V_\ell(y,0) = V_\ell(+\infty,t) = T_i, \quad y > 0, \quad t > 0 \tag{43}$$

$$e(0) = 0 \tag{44}$$

$$K_s V_{s_x}(0,t) = \frac{h_0}{\sqrt{t}}(V_s(0,t) - T_\infty), \quad t > 0 \tag{45}$$

where:

$$K_i = \rho_i k_i \text{ (new conductivity)}; \quad D_i = \frac{k_i \rho_i}{c_i} \text{ (new diffusivity)} \quad (i = s, \ell). \tag{46}$$

(b) If the coefficient $h_0$ verifies the inequality (29) we re-find the solution (30)-(32) for the free boundary problem (22)-(28).

## 3) ONE-PHASE MELTING PROBLEM WITH THE SOLOMON-WILSON-ALEXIADES MODEL OF MUSHY REGION

We consider a semi-infinite material in the solid phase at the melting temperature 0 without loss of generality. If we impose a temperature $B > 0$ at the fixed face $x = 0$, the melting process begins, and three regions can be distinguished, as follows: (Solomon – Wilson - Alexiades, Letters Heat Mass Transfer (1982); T., Int. Comm. Heat Mass Transfer (1987)):

i) the solid phase, at temperature $T = 0$, occupying the region $x > r(t),\ t > 0$;

ii) the liquid phase, at temperature $T(x,t) < 0$, occupying the region $0 < x < s(t),\ t > 0$;

iii) the mushy zone, at temperature $0$, occupying the region $s(t) < x < r(t),\ \ t > 0$. We make the following two assumptions on its structure:

a) the material in the mushy zone contains a fixed fraction $\varepsilon\ell$ (with constant $0 < \varepsilon < 1$) of the total latent heat $\ell$;

b) the width of the mushy zone is inversely proportional (with constant $\gamma > 0$) to the temperature gradient at $s(t)$.

Therefore the problem consists of finding the free boundaries $x = s(t)$ and $x = r(t)$, and the temperature $T = T(x,t)$ such that the following conditions are satisfied:

$$\rho c T_t - k T_{xx} = 0\,, \quad 0 < x < s(t)\,, \quad t > 0 \tag{47}$$

$$T\big(s(t),t\big) = 0\,, \quad t > 0 \tag{48}$$

$$-kT_x(s(t),t) = \rho \ell [(1-\varepsilon)\dot{s}(t) + \varepsilon \dot{r}(t)], \quad t > 0 \tag{49}$$

$$-T_x(s(t),t)(r(t)-s(t)) = \gamma, \quad t > 0 \tag{50}$$

$$s(0) = r(0) = 0 \tag{51}$$

$$kT_x(0,t) = -\frac{h_0}{\sqrt{t}}\big(T(0,t) - B\big), \quad t > 0 \quad (B > 0). \tag{52}$$

**Theorem 5**. If the coefficient $h_0$ verifies the inequality

$$h_0 > \frac{k}{\sqrt{\pi\alpha}} \quad \left(\alpha = \frac{k}{\rho c}\right) \tag{53}$$

there exists an instantaneous melting process and then the free boundary problem (47)-(52) has the explicit solution to a similarity type given by:

$$T(x,t) = \frac{Bh_0\sqrt{\pi\alpha}}{k} \frac{erf(\xi)}{\frac{h_0\sqrt{\pi\alpha}}{k} erf(\xi) - 1} \left[ 1 - \frac{erf(\frac{x}{2\sqrt{\alpha t}})}{erf(\xi)} \right], \quad 0 < x < s(t), \quad t > 0 \tag{54}$$

$$s(t) = 2\xi\sqrt{\alpha t} \tag{55}$$

$$r(t) = 2\mu\sqrt{\alpha t} \tag{56}$$

where

$$\mu = \xi + \frac{\gamma k}{2Bh_0\sqrt{\alpha}} W(\xi) \tag{57}$$

and $\xi > 0$ is the unique solution to the equation:

$$\frac{W(x)}{h_0}\left[x+\frac{\varepsilon\gamma k}{2B\sqrt{\alpha}}\frac{W(x)}{h_0}\right]=\frac{B}{\rho\ell\sqrt{\alpha}}, \quad x>0 \tag{58}$$

where function W is defined by:

$$W(x)=\exp(x^2)\left[\frac{h_0\sqrt{\pi\alpha}}{k}erf(x)-1\right], \quad x>0. \tag{59}$$

Moreover, we have:

$$\xi>\xi_0=erf^{-1}\left(\frac{k}{h_0\sqrt{\pi\alpha}}\right). \tag{60}$$

## DETERMINATION OF ONE UNKNOWN THERMAL COEFFICIENT WITH AN OVER-SPECIFIED CONDITION AT THE FIXED FACE

Following T., Int. Comm. Heat Mass Transfer (1987) we can obtain formula for the determination of one unknown thermal coefficient through a free boundary problem with an overspecified condition on the fixed face, i.e.: find the free boundaries $x = s(t)$ and $x = r(t)$, the temperature $T = T(x,t)$ and one unknown thermal coefficient among $\{k, c, \rho, \ell, \gamma, \varepsilon\}$ such that the following conditions are satisfied:

$$\rho c T_t - k T_{xx} = 0\,, \quad 0 < x < s(t)\,, \quad t > 0 \; (\alpha = k/\rho c) \tag{61}$$

$$T\big(s(t), t\big) = 0\,, \quad t > 0 \tag{62}$$

$$-k T_x(s(t), t) = \rho \ell [(1-\varepsilon)\dot{s}(t) + \varepsilon \dot{r}(t)], \quad t > 0 \tag{63}$$

$$-T_x(s(t), t)(r(t) - s(t)) = \gamma, \quad t > 0 \tag{64}$$

$$s(0) = r(0) = 0 \tag{65}$$

$$k T_x(0, t) = -\frac{q_0}{\sqrt{t}}, \quad t > 0. \tag{66}$$

$$k T_x(0, t) = -\frac{h_0}{\sqrt{t}}\big(T(0,t) - B\big), \quad t > 0 \quad (B > 0). \tag{67}$$

**Theorem 6.** Let $h_0$ and $q_0$ be determinated experimentally. The solution for the determination of one thermal coefficient is given by:

$$T(x,t) = \frac{q_0\sqrt{\pi\alpha}}{k} erf(\xi)\left[1 - \frac{erf(\frac{x}{2\sqrt{\alpha t}})}{erf(\xi)}\right], \quad 0 < x < s(t),\ t > 0, \tag{68}$$

$$s(t) = 2\xi\sqrt{\alpha t}, \quad t > 0, \tag{69}$$

$$r(t) = 2\mu\sqrt{\alpha t}, \quad t > 0, \tag{70}$$

and $\xi$ and the unknown thermal coefficient are computed in the summarized way in the following Table 1:

| Case # | Formulae for unknown coefficients | Parameter $\xi$ is the unique solution to the Eq. | Restrictions on data |
|---|---|---|---|
| 1 | $\xi = erf^{-1}\left(\frac{k}{\sqrt{\pi\alpha}}\left(\frac{1}{h_0}+\frac{B}{q_0}\right)\right)$ <br> $\ell = \frac{q_0}{\rho\sqrt{\alpha}}\frac{\exp(-\xi^2)}{\xi+\frac{\varepsilon\gamma k}{2q_0\sqrt{\alpha}}}$ | --------- | $\frac{k}{\sqrt{\pi\alpha}}\left(\frac{1}{h_0}+\frac{B}{q_0}\right)<1$ |
| 2 | $\xi = erf^{-1}\left(\frac{k}{\sqrt{\pi\alpha}}\left(\frac{1}{h_0}+\frac{B}{q_0}\right)\right)$ <br> $\gamma = \frac{2q_0\sqrt{\alpha}}{\varepsilon k}\mathrm{e}^{-\xi^2}\left[\frac{q_0}{\rho\ell\sqrt{\alpha}}\mathrm{e}^{-\xi^2}-\xi\right]$ | --------- | $\frac{k}{\sqrt{\pi\alpha}}\left(\frac{1}{h_0}+\frac{B}{q_0}\right)<1$ <br> $\ell < \frac{q_0}{\rho\sqrt{\alpha}}\frac{1}{g_1\left(erf^{-1}\left(\frac{k}{\sqrt{\pi\alpha}}\left(\frac{1}{h_0}+\frac{B}{q_0}\right)\right)\right)}$ <br> where $g_1(x)=x\exp(x^2)$ |

| | | | |
|---|---|---|---|
| 3 | $\xi = erf^{-1}\left(\frac{k}{\sqrt{\pi\alpha}}\left(\frac{1}{h_0}+\frac{B}{q_0}\right)\right)$ <br> $\varepsilon = \frac{2q_0\sqrt{\alpha}}{\gamma k}\mathrm{e}^{-2\xi^2}\left[\frac{q_0}{\rho\ell\sqrt{\alpha}}\mathrm{e}^{-\xi^2} - g_1(\xi)\right]$ | --------- | $\frac{k}{\sqrt{\pi\alpha}}\left(\frac{1}{h_0}+\frac{B}{q_0}\right) < 1$ <br> $\frac{1}{g_1(\xi)+\frac{\gamma k}{2q_0\sqrt{\alpha}}e^{2\xi^2}} < \frac{\rho\sqrt{\alpha}}{q_0}\ell < \frac{1}{g_1(\xi)}$ |
| 4 | $k = \frac{\pi}{\rho c\left(\frac{1}{h_0}+\frac{B}{q_0}\right)^2} erf^2(\xi)$ | Eq. (A) | ------------------ |
| 5 | $\rho = \frac{\pi}{kc\left(\frac{1}{h_0}+\frac{B}{q_0}\right)^2} erf^2(\xi)$ | Eq. (A) | ------------------ |
| 6 | $c = \frac{\pi}{\rho k\left(\frac{1}{h_0}+\frac{B}{q_0}\right)^2} erf^2(\xi)$ | Eq. (B) | $\frac{\varepsilon\gamma\ell\rho k}{2q_0^2} > 1$ |

Table 1 Summary of the determination of one thermal coefficient through a one-phase Lamé-Clapeyron-Stefan problem with an overspecfied condition on the fixed face (6 cases)

where Eqs. (A) and (B) are defined by:

(A): $$g_1(x)\left[x+\frac{\varepsilon\gamma\sqrt{\pi}}{2q_0\left(\frac{1}{h_0}+\frac{B}{q_0}\right)}erf(x)\right]=\frac{q_0 c}{\ell\sqrt{\pi}}\left(\frac{1}{h_0}+\frac{B}{q_0}\right),\quad x>0 \qquad (71)$$

(B): $$g_1(x)+\frac{\sqrt{\pi}}{\frac{1}{h_0}+\frac{B}{q_0}}erf(x)\left[\frac{\varepsilon\gamma}{2q_0}e^{2x^2}-\frac{q_0}{k\rho\ell}\right]=0,\quad x>0 \qquad (72)$$

## 4) TWO-PHASE SOLIDIFICATION PROBLEM WITH A MUSHY ZONE MODEL

We consider a semi-infinite material initially in the liquid phase at the temperature $T_i > T_f = 0$. If we impose a temperature $-D < T_f = 0$ at the fixed face $x = 0$, the solidification process begins, and three regions can be distinguished, as follows: (Solomon – Wilson - Alexiades, Letters Heat Mass Transfer (1982); T., Comput. Appl. Math. (1990)):

i) the solid phase, at temperature $T_s = T_s(x,t) < 0$, occupying the region $0 < x \le s(t), \ t > 0$;

ii) the liquid phase, at temperature $T_\ell = T_\ell(x,t) > 0$, occupying the region $x > r(t), \quad t > 0$;

iii) the mushy zone, at temperature $T_f = 0$, occupying the region $s(t) < x < r(t), \quad t > 0$. We make the following two assumptions on its structure:

a) the material in the mushy zone contains a fixed fraction $\varepsilon\ell$ (with constant $0 < \varepsilon < 1$) of the total latent heat $\ell$;

b) the width of the mushy zone is inversely proportional (with constant $\gamma > 0$) to the temperature gradient at $s(t)$.

Therefore, the problem consists of finding the two free boundaries $x = s(t), x = r(t)$, and the temperature:

$$T(x,t) = \begin{cases} T_s(x,t) > 0 & \text{if } 0 < x < s(t),\, t > 0 \\ 0 & \text{if } s(t) \le x \le r(t),\, t > 0 \\ T_\ell(x,t) < 0 & \text{if } r(t) < x,\, t > 0 \end{cases} \tag{73}$$

defined for $x > 0$ and $t > 0$, such that the following conditions are satisfied:

$$\alpha_s T_{s_{xx}} = T_{s_t}, \qquad 0 < x < s(t),\, t > 0 \tag{74}$$

$$\alpha_\ell T_{\ell_{xx}} = T_{\ell_t}, \qquad r(t) < x,\, t > 0 \tag{75}$$

$$s(0) = r(0) = 0, \tag{76}$$

$$T_\ell(r(t),t) = T_s(s(t),t) = 0, \quad t > 0 \tag{77}$$

$$k_s T_{s_x}(s(t),t) - k_\ell T_{\ell_x}(r(t),t) = \rho\ell[\varepsilon\dot{s}(t) + (1-\varepsilon)\dot{r}(t)], \quad t > 0, \tag{78}$$

$$T_{s_x}(s(t),t)\,(r(t) - s(t)) = \gamma, \quad t > 0 \tag{79}$$

$$T_\ell(x,0) = T_\ell(+\infty,t) = T_i, \quad x > 0,\, t > 0 \tag{80}$$

$$k_s T_{s_x}(0,t) = \frac{h_0}{\sqrt{t}}\left(T_s(0,t) + D\right), \quad t > 0 \tag{81}$$

**Theorem 7** If the coefficient $h_0$ verifies the inequality

$$h_0 > \frac{\gamma k_s}{2D\eta_0\sqrt{\alpha_\ell}} \tag{82}$$

where $\eta_0 = \eta_0\left(\frac{\gamma k_s}{T_i k_\ell}, \frac{(1-\varepsilon)\ell}{T_i c_\ell}\right) > 0$ is the unique solution of the equation:

$$F_1(x) = F_4(x), \quad x > 0 \tag{83}$$

where functions $F_1$ and $F_4$ is given by:

$$F_1(x) = \frac{e^{-x^2}}{erfc(x)}, \quad F_4(x) = \frac{\gamma k_s\sqrt{\pi}}{2T_i k_\ell}\frac{1}{x} - \frac{(1-\varepsilon)\ell\sqrt{\pi}}{T_i c_\ell}x, \quad x > 0 \tag{84}$$

then there exists an instantaneous solidification process and therefore the free boundary problem (74)-(81) has the explicit solution to a similarity type given by:

$$T_s(x,t) = \frac{-D\,erf(\xi)}{erf(\xi) + \frac{k_s}{h_0\sqrt{\pi\alpha_s}}}\left[1 - \frac{erf\left(\frac{x}{2\sqrt{\alpha_s t}}\right)}{erf(\xi)}\right], \quad 0 < x < s(t), \quad t > 0 \tag{85}$$

$$T_\ell(x,t) = \frac{T_i\ erf(\mu)}{erfc(\mu)}\left[\frac{erf\left(\frac{x}{2\sqrt{\alpha_\ell t}}\right)}{erf(\mu)} - 1\right], \quad x > r(t), \quad t > 0 \tag{86}$$

$$s(t) = 2\xi\sqrt{\alpha_s t} \tag{87}$$

$$r(t) = 2\mu\sqrt{\alpha_\ell t} \tag{88}$$

where

$$\mu = \sqrt{\frac{\alpha_s}{\alpha_\ell}}\, W_1(\xi) \tag{89}$$

and $\xi > 0$ is the unique solution to the equation:

$$F(x) = G(x), \quad x > 0 \tag{90}$$

where functions $W_1, G, F_3$ and F are defined by the followin expressions:

$$W_1(x) = x + \frac{\gamma\sqrt{\pi}}{2D} \mathrm{e}^{x^2} \left[ erf(x) + \frac{k_s}{h_0\sqrt{\pi\alpha_s}} \right], \quad x > 0 \tag{91}$$

$$G(x) = \varepsilon x + (1-\varepsilon) W_1(x), \quad x > 0 \tag{92}$$

$$F_3(x) = \frac{e^{-x^2}}{erf(x) + \frac{k_s}{h_0\sqrt{\pi\alpha_s}}}, \quad x > 0 \tag{92}$$

$$F(x) = \frac{Dc_s}{\ell\sqrt{\pi}} F_3(x) - \frac{T_i}{\ell}\sqrt{\frac{k_\ell c_s c_\ell}{k_s \pi}} F_1\left( \sqrt{\frac{\alpha_s}{\alpha_\ell}} W_1(x) \right), \quad x > 0\,. \tag{93}$$

## 5) RUBINSTEIN BINARY-ALLOY SOLIDIFICATION PROBLEM

A semi-infinite material of a binary alloy consisting of two components $A$ and $B$ is considered. Let $C$ and $T$ the concentration and the temperature of the component $B$. We suppose that the solidification of the alloy is governed by a graph of balance of phase consisting of a curve "**liquidus**" $C = f_\ell(T)$ and a curve "**solidus**" $C = f_s(T)$. We assume that $f_s$ and $f_\ell$ are increasing functions in the variable T with:

$$f_\ell(T_A) = f_s(T_A) < f_\ell(T) < f_s(T) < f_\ell(T_B) = f_s(T_B), \quad T_A < T < T_B, \tag{94}$$

where $T_A$ and $T_B$ are the temperatures of merger of $A$ and $B$ respectively. We also assume that the material is in the solid phase if $C > f_s(T)$ and in the liquid phase if $C < f_\ell(T)$.

When the concentration $C$ is between $f_s(T)$ and $f_\ell(T)$ the state of the material is not well definied and it is known by mushy region according to the description of the model proposed by:

- Rubinstein, AMS (1971),
- Solomon – Wilson - Alexiades, Quart. Appl. Math. (1983)

which can by appreciated in the Figure 1.

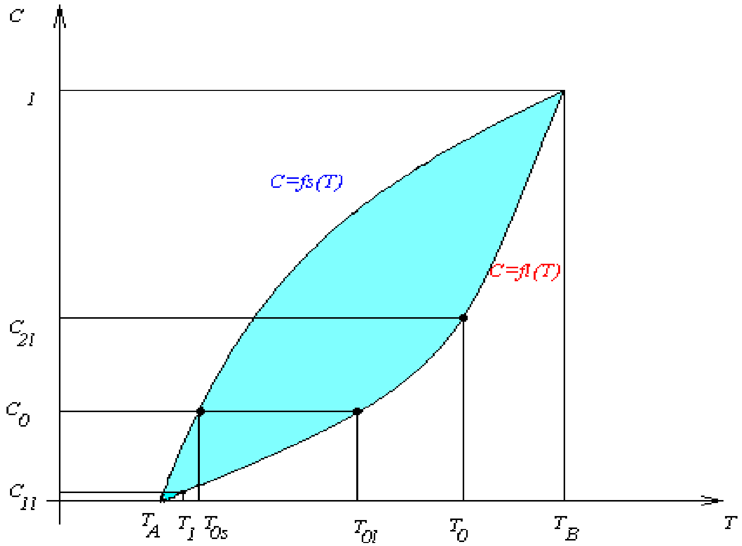


Fig.1: Concentration vs. Temperature (graph of phase balance with liquidus and solidus curves)

We suppose that the alloy is initially in liquid phase at the constant temperature $T_0$ and at the constant concentration $C_0$. Then a boundary condition is imposed on the fixed face $x = 0$ and a front of solidification starts by separating instantaneously the solid phase $\left(x < s(t)\right)$ from the liquid phase $\left(x > s(t)\right)$.The mathematical formulation of this process of crystallization consists in finding the temperature $T = T(x,t)$ and the concentration $C = C(x,t)$, both defined for $x > 0$ and $t > 0$, the free boundary $x = s(t)$, defined for $t > 0$, and the critical temperature of solidification $T_k$ so that the following conditions are verified:

$$\alpha_s T_{s_{xx}} = T_{s_t}, \quad 0 < x < s(t), \quad t > 0 \tag{95}$$

$$\alpha_\ell T_{\ell_{xx}} = T_{\ell_t}, \quad s(t) < x, \quad t > 0 \tag{96}$$

$$d_s C_{s_{xx}} = C_{s_t}, \quad 0 < x < s(t), \quad t > 0 \tag{97}$$

$$d_\ell C_{\ell_{xx}} = C_{\ell_t}, \quad x > s(t), \quad t > 0 \tag{98}$$

$$T_\ell(x,0) = T_\ell(\infty,t) = T_0, \quad \text{with } T_A < T_0 < T_B, \quad \text{x>0} \tag{99}$$

$$C_l(x,0) = C_0, \quad x > 0 \tag{100}$$

$$T_s(s(t),t) = T_\ell(s(t),t) = T_k, \quad t > 0 \tag{101}$$

$$C_s(s(t),t) = f_s(T_k), \quad t > 0 \tag{102}$$

$$C_\ell(s(t),t) = f_\ell(T_k), \quad t > 0 \tag{103}$$

$$k_s T_{s_x}(s(t),t) - k_\ell T_{\ell_x}(s(t),t) = \rho \ell \dot{s}(t), \quad t > 0 \tag{104}$$

$$d_\ell C_{\ell_x}(s(t),t) - d_s C_{s_x}(s(t),t) = \left[ f_s\left(T_k\right) - f_\ell(T_k) \right] \dot{s}(t), \quad t > 0 \tag{105}$$

$$C_{s_x}(0,t) = 0, \quad t > 0 \tag{106}$$

and the boundary condition on $x = 0$ given by the convective condition:

$$k_s T_{s_x}(0,t) = \frac{h_0}{\sqrt{t}}(T_s(0,t) - T_1), \quad t > 0 \quad \left(h_0 > 0\right) \tag{107}$$

or the heat flux condition:

$$k_s T_{s_x}(0,t) = \frac{q_0}{\sqrt{t}}, \quad t > 0 \quad \left(q_0 > 0\right) \tag{108}$$

**Theorem 8** If $h_0$ verifies the inequalities:

$$\frac{(T_0 - T_{0_\ell})k_\ell}{(T_{0_\ell} - T_1)\sqrt{\pi\alpha_\ell}} < h_0 < \frac{(T_0 - T_{0_s})k_\ell}{(T_{0_s} - T_1)\sqrt{\pi\alpha_\ell}} \tag{109}$$

where $T_{0_\ell} = f_\ell^{-1}(C_0)$ y $T_{0s} = f_s^{-1}(C_0)$ then there exists an instantaneous unique solution of the similarity type for the free boundary problem (95)-(107).

**Theorem 9** If $q_0$ verifies the following inequalities

$$\frac{(T_0 - T_{0_\ell})k_l}{\sqrt{\pi\alpha_\ell}} < q_0 < \frac{(T_0 - T_{0_s})k_\ell}{\sqrt{\pi\alpha_\ell}} \tag{110}$$

then there exists a unique solution of the similarity type for the free boundary problem (95)-(106) and (108).

**6) THAWING IN A SATURATED POROUS MEDIUM BY CONSIDERING A DENSITY JUMP AND THE INFLUENCE OF THE PRESSURE ON THE MELTING TEMPERATURE (In Progress).**

We consider the problem of thawing of a partialIy frozen porous medium, saturated with an incompressible liquid. For a detailed exposition of the physical background we refer to:

- Charach - Rubinstein, J. Appl. Phys. (1992);
- Fasano – Guan – Primicerio – Rubinstein, Meccanica (1993);
- Nakano, Cold. Reg. Sci. Tech. (1990);
- O'Neill - Miller, Water Resour. Res. (1985);
- Talamucci, Survey Math. Industry (1997);
- Fasano – Primicerio – T., Math. Models Meth. Appl. Sci. (1999) for $\left(u(0,t)=B>0\right)$
- Lombardi – T., Meccanica (2002) for $\left(k_U u_x(0,t)=-\dfrac{q_0}{\sqrt{t}}>0\right)$

More specifically, we deal with the following situations:

(i) a sharp interface between the frozen part and the unfrozen part of the domain exists (sharp, in the macroscopic sense);

(ii) the frozen phase is at rest with respect to the porous skeleton, which will be considered to be undeformable;

(iii) due to the density jump between the liquid and solid phases, thawing can induce either desaturation or water movement in the melting regíon. We will consider the latter situation, assuming that liquid is continuously supplied to keep the medium saturated.

The unknowns of the problem are the function $x=s(t)$, representing the free boundary, and the two functions $u(x, t)$ and $v(x, t)$ representing the temperature of the unfrozen and of the frozen zone respectively which must satisfy the following conditions:

$$u_t = a_1 u_{xx} - b\rho \dot{s}(t) u_x, \qquad 0 < x < s(t), \quad t > 0 \tag{111}$$

$$v_t = a_2 v_{xx}, \qquad x > s(t), \quad t > 0 \tag{112}$$

$$u(s(t),t) = v(s(t),t) = d\rho\, s(t)\,\dot{s}(t), \quad t > 0 \tag{113}$$

$$k_F\, v_x(s(t),t) - k_U\, u_x(s(t),t) = \alpha\,\dot{s}(t) + \beta\,\rho\, s(t)\ (\dot{s}(t))^2, \quad t > 0 \tag{114}$$

$$v(x,0) = v(+\infty,t) = -A < 0, \quad x > 0,\, t > 0 \tag{115}$$

$$s(0) = 0 \tag{116}$$

$$k_U u_x(0,t) = \frac{h_0}{\sqrt{t}}\big(u(0,t) - B\big), > 0\,. \tag{117}$$

with

$$a_1 = \alpha_1^2 = \frac{k_U}{\rho_U c_U}, \quad a_2 = \alpha_2^2 = \frac{k_F}{\rho_F c_F}, \quad b = \frac{\varepsilon \rho_W c_W}{\rho_U c_U}, \quad d = \frac{\varepsilon \gamma \mu}{K}$$
$$\rho = \frac{\rho_W - \rho_I}{\rho_W}, \quad \alpha = \varepsilon \rho_I \ell, \quad \beta = \frac{\varepsilon^2 \rho_I \gamma \mu (c_W - c_I)}{K} = \varepsilon d \rho_I (c_W - c_I) \tag{118}$$

where:

$\varepsilon$: porosity,
$\rho_W$ and $\rho_I$: density of water and ice,
$c$: specific heat at constant density,
$k_U$ and $k_F$: conductivity of the unfrozen and frozen zones,
$u = v = 0$: the melting point at atmospheric pressure,
$\ell$: latent heat at $u = 0$,
$\gamma$: coefficient in the Clausius-Clapeyron law,
$\mu > 0$: viscosity of liquid,
$K > 0$: hydraulic permeability,
$B > 0$: external temperature at the fixed face $x = 0$,
$-A < 0$: initial temperature,
$q_0 > 0$: coefficient which characterizes de heat flux at the fixed face $x = 0$,
$h_0 > 0$: coefficient which characterizes de heat transfer at the fixed face $x = 0$.

**Theorem 10** The free boundary problem (111) – (117) has the similarity solution

$$s(t) = 2\xi\alpha_1\sqrt{t}, \tag{119}$$

$$u(x,t) = \frac{Bg(p,\xi) + \dfrac{mk_U}{2h_0\alpha_1}\xi^2 + (m\xi^2 - B)\displaystyle\int_0^{x/(2\alpha_1\sqrt{t})} \exp(-r^2 + pr\xi)\,dr}{g(p,\xi) + \dfrac{k_U}{2h_0\alpha_1}} \tag{120}$$

$$v(x,t) = \frac{m\xi^2 + A\,erf(\gamma_0\xi) - (A + m\xi^2)erf\left(\dfrac{x}{2\alpha_2\sqrt{t}}\right)}{erfc(\gamma_0\xi)} \tag{121}$$

if and only if the coefficient $\xi > 0$ satisfies the following equation:

$$\delta_1(1 - \frac{AM}{B}y^2)\,G_1(p,y) - \delta_2\,G_2(M,y) = y + Ny^3, \quad y > 0, \tag{122}$$

where

$$g(p,y)=\int_0^y \exp(-r^2+pyr)\,dr\,, \tag{169}$$

$$G_1(p,y)=\frac{\exp((p-1)y^2)}{K_0+g(p,y)}, \qquad G_2(M,y)=(1+My^2)\frac{\exp(-\gamma_0^2 y^2)}{erfc(\gamma_0 y)} \tag{170}$$

and the constants are defined as follows:

$$K_0=\frac{k_U}{2\alpha_1 h_0}>0, \quad K_2=\frac{k_F}{\alpha_2\sqrt{\pi}}>0, \quad \gamma_0=\frac{\alpha_1}{\alpha_2}>0, \quad \delta=\alpha\alpha_1>0 \tag{171}$$

$$m=2d\rho\alpha_1^2, \quad \upsilon=2\beta\rho\alpha_1^3, \quad \delta_1=\frac{k_U B}{2\delta\alpha_1}>0, \quad \delta_2=\frac{K_2 A}{\delta}>0 \tag{172}$$

$$N=\frac{\upsilon}{\delta}\in\mathbb{R}, \quad M=\frac{2d\rho\alpha_1^2}{A}\in\mathbb{R}, \quad p=2b\rho\in\mathbb{R}. \tag{173}$$

Moreover, the existence and uniqueness of the unknown coefficient $\xi>0$ depends on the sign of the three dimensionless real parameters $p$, $M$ and $N$ of the problem.

**Remark:**

Other related free boundary problems with explicit solutions with a convective boundary conditions at the fixed face $x = 0$ are:

- One-dimensional two-phase with either shrinkage or expansion:
  Natale – Santillan Marcus – T., Nonlinear Anal. Real World Appl. (2010)

- One-dimensional one-phase non-classical Stefan problem:
  Briozzo – T., Int. J. Diff. Eq. (2010).

**THANK YOU VERY MUCH**

**FOR YOUR ATTENTION**