Weak formulations of PDEs in thermomechanics

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- The most recent applications:
  - ◇ The liquid crystal flows [joint works with: E. Feireisl, G. Schimperna (Nonlinearity, 2011), E. Feireisl, M. Frémond, G. Schimperna, (ARMA, to appear), E. Feireisl, G. Schimperna, A. Zarnescu (in preparation)]
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- The main advantages and the potential future perspectives:
  - ◇ Non-isothermal mixtures of binary immiscible fluids [with S. Frigeri, G. Schimperna, ...]
  - ◇ The induction hardening of steel [with D. Hömberg], the SMA with possibility of voids [with M. Frémond], ...

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- Damage phenomena:
  - **aim:** deal with diffuse interface models in thermoviscoelasticity accounting for
    - the evolution of the displacement variables
    - the temperature
    - the order (damage) parameter χ

    where the momentum equation for contains χ-dependent elliptic operators, which may degenerate at the *pure phases*

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- Liquid crystals

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\begin{aligned}
&\operatorname{div}\mathbf{v}=0, \quad \mathbf{v}_t+\mathbf{v}\cdot\nabla_x\mathbf{v}+\nabla_x p=\operatorname{div}\mathbb{S}+\operatorname{div}\sigma^{nd}+\mathbf{g}\\
&\mathbb{S}=\nu(\theta)\left(\nabla_x\mathbf{v}+\nabla_x^t\mathbf{v}\right), \quad \sigma^{nd}=-\nabla_x\mathbf{d}\odot\nabla_x\mathbf{d}+(\partial_{\mathbf{d}}W(\mathbf{d})-\Delta\mathbf{d})\otimes\mathbf{d}\\
&\theta_t+\mathbf{v}\cdot\theta+\operatorname{div}\mathbf{q}=\mathbb{S}:\nabla_x\mathbf{v}+|\Delta\mathbf{d}-\partial_{\mathbf{d}}W(\mathbf{d})|^2\\
&\mathbf{d}_t+\mathbf{v}\cdot\nabla_x\mathbf{d}-\mathbf{d}\cdot\nabla_x\mathbf{v}=\Delta\mathbf{d}-\partial_{\mathbf{d}}W(\mathbf{d})
\end{aligned}
$$

- Damage

$$
\begin{aligned}
&c(\theta)\theta_t+\chi_t\theta-\rho\theta\operatorname{div}\mathbf{u}_t-\operatorname{div}(k(\theta)\nabla\theta))=g\\
&\mathbf{u}_{tt}-\operatorname{div}(a(\chi)\varepsilon(\mathbf{u}_t)+b(\chi)\varepsilon(\mathbf{u})-\rho\theta\mathbf{1})=\mathbf{f}\\
&\chi_t+\partial I_{(-\infty,0]}(\chi_t)+A_s\chi+W'(\chi)\ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2}+\theta
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2. a *generalization of the principle of virtual powers* inspired by:

   2.1. the notion of *energetic solution* - A. Mielke and co-authors ([Bouchitté, Mielke, Roubíček, ZAMP. Angew. Math. Phys. (2009) and [Mielke, Roubíček, Zeman, Comput. Methods Appl. Mech. Engrg. (2010)]) for rate-independent processes for damage phenomena and

   2.2. a notion of *weak solution* introduced by [Heinemann, Kraus, WIAS preprint 1569 and WIAS preprint 1520, to appear on Adv. Math. Sci. Appl. (2010)] for non-degenerating isothermal diffuse interface models for phase separation and damage

Entropic formulation: a phase transitions model

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We consider there a model for solid-liquid phase transitions associated to a nonlinear PDE system

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$\Longrightarrow$ a new notion of solution is needed

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Finally, couple these relations to a suitable phase dynamics.

The entropy production

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$$\int_0^T \int_\Omega s_t \varphi - \int_0^T \int_\Omega \frac{\mathbf{q}}{\theta} \cdot \nabla \varphi = \int_0^T \int_\Omega r \varphi \quad \forall \varphi \in \mathcal{D}(\overline{Q}_T),$$

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r represents the entropy production rate. Then, in order to comply with the Clausius-Duhem inequality, we assume:

(i) r is a nonnegative measure on $[0, T] \times \overline{\Omega} =: \overline{Q}_T$;

(ii) $r \geq \frac{1}{\theta}\left(|\chi_t|^2 - \frac{\mathbf{q} \cdot \nabla \theta}{\theta}\right) \geq 0.$

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Taking $\mathbf{q} = -\nabla \theta$, $s = \log \theta + \chi$, we get

$$\int_0^T \int_\Omega \Big((\log \theta + \chi)\, \partial_t \varphi - \nabla \log \theta \cdot \nabla \varphi \Big) dxdt \leq \int_0^T \int_\Omega \frac{1}{\theta} \left(-|\chi_t|^2 - \nabla \log \theta \cdot \nabla \theta \right) \varphi dxdt$$

for every test function $\varphi \in \mathcal{D}(\overline{Q}_T)$, $\varphi \geq 0$

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for every test function $\varphi \in \mathcal{D}(\overline{Q}_T)$, $\varphi \geq 0$

$\Rightarrow$ the total entropy is controlled by dissipation.

The energy conservation and phase relation

The total energy has to be preserved. Hence

$$E(t) = E(0) \text{ for a.e. } t \in [0, T],$$

where

$$E \equiv \int_\Omega \left(\theta + W(\chi) + \frac{|\nabla \chi|^2}{2} \right) dx \, .$$

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Finally, the phase dynamics results as

$$\chi_t - \Delta\chi + W'(\chi) = \theta - \theta_c \quad \text{a.e. in } \Omega \times (0, T),$$

where W is a double well or double obstacle potential: $W = \widehat{\beta} + \widehat{\gamma}$ where

$\widehat{\beta} : \mathbb{R} \to [0, +\infty]$ is proper, lower semi-continuous, convex function

$\widehat{\gamma} \in C^2(\mathbb{R})$, $\widehat{\gamma}' \in C^{0,1}(\mathbb{R})$: $\widehat{\gamma}''(r) \geq -K$ for all $r \in \mathbb{R}$, $W(r) \geq c_w r^2$ for all $r \in \mathrm{dom}(\widehat{\beta})$

Examples: $\widehat{\beta}(r) = r\ln(r) + (1 - r)\ln(1 - r)$ or $\widehat{\beta}(r) = I_{[0,1]}(r)$.

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- However, in this case and similarly in many other situations, to prove that the solution has this extra regularity is **out of reach**
- **It can be suitable also in different applications** such as the ones related to phase transitions in viscoelastic materials, SMA, liquid crystal flows, etc.

Entropic formulation: the hydrodynamics of liquid crystal flows

A recent application: non-isothermal liquid crystals

- The motivations:
  - Theoretical studies of these types of materials are motivated by **real-world applications**: proper functioning of many practical devices relies on optical properties of certain liquid crystalline substances in the presence or absence of an electric field: **a multi-billion dollar industry**
  - At the molecular level, what marks the difference between a liquid crystal and an ordinary, isotropic fluid is that, while the centers of mass of LC molecules do not exhibit any long-range correlation, **molecular orientations do exhibit orientational correlations**

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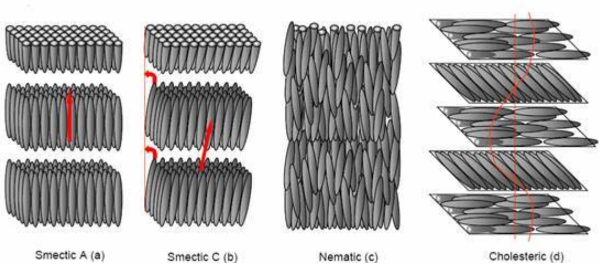
- Our most recent results:
  1. E. Feireisl, M. Frémond, E. R., G. Schimperna, A new approach to non-isothermal models for nematic liquid crystals, ARMA to appear, preprint arXiv:1104.1339v1 (2011)
  2. E. Feireisl, E.R., G. Schimperna, A. Zarnescu, Evolution of non-isothermal Landau-de Gennes nematic liquid crystals flows, paper in preparation

To the present state of knowledge, three main types of liquid crystals are distinguished, termed *smectic*, **nematic** and *cholesteric*



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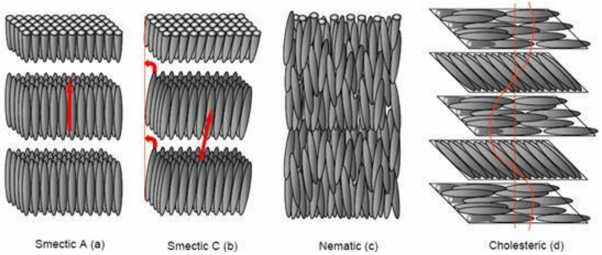

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The *smectic* phase forms well-defined layers that can slide one over another in a manner very similar to that of a soap

The **nematic** phase: the molecules have long-range orientational order, but no tendency to the formation of layers. Their center of mass positions all point in the same direction (within each specific domain)

Crystals in the *cholesteric* phase exhibit a twisting of the molecules perpendicular to the director, with the molecular axis parallel to the director

Our main aim

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- Most mathematical work has been done on the Oseen-Frank theory, in which the mean orientation of the rod-like molecules is described by a **vector field d**. However, more popular among physicists is the Landau-de Gennes theory, in which the order parameter describing the orientation of molecules is a matrix, the so-called **$\mathbb{Q}$-tensor**

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- The flow **velocity u** evidently disturbs the alignment of the molecules and also the converse is true: a change in the alignment will produce a perturbation of the velocity field $\mathbf{u}$. Moreover, we want to include in our model also the **changes of the temperature θ**

The Landau-de Gennes theory: the molecular orientation

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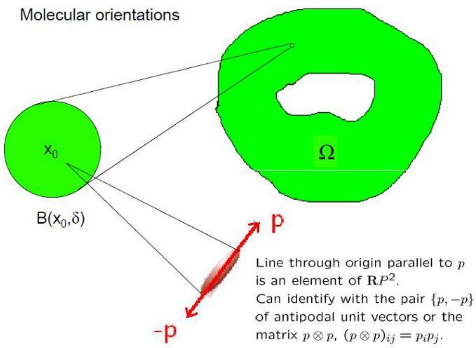
- Consider a nematic liquid crystal filling a bounded connected container Ω in $\mathbb{R}^3$ with "regular" boundary
- The distribution of molecular orientations in a ball $B(x_0, \delta)$, $x_0 \in \Omega$ can be represented as a probability measure μ on the unit sphere $\mathbb{S}^2$ satisfying $\mu(E) = \mu(-E)$ for $E \subset \mathbb{S}^2$
- For a continuously distributed measure we have $d\mu(p) = \rho(p)dp$ where dp is an element of the surface area on $\mathbb{S}^2$ and $\rho \geq 0$, $\int_{\mathbb{S}^2} \rho(p)dp = 1$, $\rho(p) = \rho(-p)$



The Landau-de Gennes theory: the $\mathbb{Q}$-tensor

- The first moment $\int_{\mathbb{S}^2} p \, d\mu(p) = 0$, the second moment $M = \int_{\mathbb{S}^2} p \otimes p \, d\mu(p)$ is a symmetric non-negative 3×3 matrix (for every $\mathbf{v} \in \mathbb{S}^2$,
$\mathbf{v} \cdot M \cdot \mathbf{v} = \int_{\mathbb{S}^2} (\mathbf{v} \cdot p)^2 \, d\mu(p) = < \cos^2 \theta >$, where θ is the angle between p and $\mathbf{v}$) satisfying $\mathrm{tr}(M) = 1$

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- If the orientation of molecules is equally distributed in all directions (the distribution is *isotropic*) and then $\mu = \mu_0$, where $d\mu_0(p) = \frac{1}{4\pi} dS$. In this case the second moment tensor is $M_0 = \frac{1}{4\pi} \int_{\mathbb{S}^2} p \otimes p\, dS = \frac{1}{3}\mathbf{1}$, because $\int_{\mathbb{S}^2} p_1 p_2\, dS = 0$, $\int_{\mathbb{S}^2} p_1^2\, dS = \int_{\mathbb{S}^2} p_2^2\, dS$, etc., and $\mathrm{tr}(M_0) = 1$

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▶ **The de Gennes $\mathbb{Q}$-tensor** measures the deviation of M from its isotropic value

$$\mathbb{Q} = M - M_0 = \int_{\mathbb{S}^2} \left(p \otimes p - \frac{1}{3}\mathbf{1} \right) d\mu(p)$$

▶ Note that (cf. [Ball, Majumdar, Molecular Crystals and Liquid Crystals (2010)])

1. $\mathbb{Q} = \mathbb{Q}^T$
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1.+2. implies $\mathbb{Q} = \lambda_1 \mathbf{n}_1 \otimes \mathbf{n}_1 + \lambda_2 \mathbf{n}_2 \otimes \mathbf{n}_2 + \lambda_3 \mathbf{n}_3 \otimes \mathbf{n}_3$, where $\{\mathbf{n}_i\}$ is an othonormal basis of eigenvectors of $\mathbb{Q}$ with corresponding eigenvalues λ_i such that $\lambda_1 + \lambda_2 + \lambda_3 = 0$

2.+3. implies $-\frac{1}{3} \leq \lambda_i \leq \frac{2}{3}$

- $\mathbb{Q} = 0$ does not imply $\mu = \mu_0$ (e.g. $\mu = \frac{1}{6} \sum_{i=1}^{3} (\delta_{e_i} + \delta_{-e_i})$)

The reduction to the Oseen-Frank model

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Reduction to the Oseen-Frank (1925, 1952) model (Ericksen model, 1991): the *uniaxial case*: $\lambda_1 = \lambda_2 = -\frac{s}{3}$, $\lambda_3 = \frac{2s}{3}$, setting $\mathbf{n}_3 = \mathbf{d}$ where $\mathbf{n}_i$ is an orthonormal basis of eigenvectors of $\mathbb{Q}$ corresponding to λ_i, we have

$$\mathbb{Q} = -\frac{s}{3}(\mathbf{1} - \mathbf{d} \otimes \mathbf{d}) + \frac{2s}{3}\mathbf{d} \otimes \mathbf{d} = s\left(\mathbf{d} \otimes \mathbf{d} - \frac{1}{3}\mathbf{1}\right),$$

where $-\frac{1}{2} \le s \le 1$.
Here $s \in \mathbb{R}$ is a real scalar order parameter that measures the degree of orientational ordering and **d is a vector** representing the direction of preferred molecular alignment: the **director field**.

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Suppose (for the moment) that the material is incompressible, homogeneous and at a constant temperature T in Ω. At each $x \in \Omega$ we have an order parameter tensor $\mathbb{Q}(x)$ and **the Landau-de Gennes free energy** (defined in the space of traceless symmetric 3×3 matrixes) is

$$\mathcal{F}_{LG}(\mathbb{Q}) = \int_\Omega \left(\frac{L}{2} |\nabla \mathbb{Q}(x)|^2 + f_B(\mathbb{Q}(x)) \right) \, dx \,,$$

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where

- $|\nabla \mathbb{Q}|^2 = \sum_{i,j,k=1}^3 \mathbb{Q}_{ij,k} \mathbb{Q}_{ij,k}$ is the elastic energy density that penalizes spatial inhomogeneities and $L > 0$ is a material-dependent elastic constant
- $f_B(\mathbb{Q})$ **is the bulk free energy density**, e.g., (following [de Gennes, Prost (1995)])

$$f_B(\mathbb{Q}) = \frac{\alpha(T - T^*)}{2} \mathrm{tr}(\mathbb{Q}^2) - \frac{b}{3} \mathrm{tr}(\mathbb{Q}^3) + \frac{c}{4} (\mathrm{tr}(\mathbb{Q}^2))^2$$

  where α, b, c are material-dependent positive constants, T is the absolute temperature and T^* is a characteristic liquid crystal temperature. Call $a = \alpha(T - T^*)$

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  it is reasonable to consider a theory where $\mathbb{Q}$ is required to be uniaxial with constant scalar order parameter $s > 0$, i.e.

$$\mathbb{Q} = s \left(\mathbf{d} \otimes \mathbf{d} - \frac{1}{3} \mathbf{1} \right) \,.$$

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- In this case f_B is constant and we can consider only the elastic energy and calculating it in terms of $\mathbf{d}$ we obtain the simplest form of the **Oseen-Frank free energy** (1925, 1958)

$$\mathcal{F}_{OF} = L s^2 \int_\Omega |\nabla \mathbf{d}(x)|^2 \, dx$$

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- In the Landau-de Gennes free energy there is no a-priori bound on the eigenvalues
- In order to naturally enforce the physical constraints in the eigenvalues of the symmetric, traceless tensors $\mathbb{Q}$, Ball and Majumdar have recently introduced in [Ball, Majumdar, Molecular Crystals and Liquid Crystals (2010)] a singular component

$$f(\mathbb{Q}) = \begin{cases} \inf_{\rho \in \mathcal{A}_{\mathbb{Q}}} \int_{S^2} \rho(\mathbf{p}) \log(\rho(\mathbf{p}))\, \mathrm{d}\mathbf{p} \text{ if } \lambda_i[\mathbb{Q}] \in (-1/3, 2/3), \ i = 1,2,3, \\ \\ \infty \text{ otherwise,} \end{cases}$$

$$\mathcal{A}_{\mathbb{Q}} = \left\{ \rho : S^2 \to [0,\infty) \;\Big|\; \int_{S^2} \rho(\mathbf{p})\, \mathrm{d}\mathbf{p} = 1; \mathbb{Q} = \int_{S^2} \left(\mathbf{p} \otimes \mathbf{p} - \frac{1}{3}\mathbb{I} \right) \rho(\mathbf{p})\, \mathrm{d}\mathbf{p} \right\}.$$

to the bulk free-energy f_B enforcing the eigenvalues to stay in the interval $(-\frac{1}{3}, \frac{2}{3})$.

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⇒ For the **Landau-de Gennes** free energy with "regular" potential, the hydrodynamic theory has been developed in [Paicu, Zarnescu, SIAM (2011) and ARMA (2012)]

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$\Longrightarrow$ The proposed model is shown compatible with *First and Second laws* of thermodynamics, and the existence of **global-in-time weak solutions** for the resulting PDE system is established, without any essential restriction on the size of the data, or on the space dimension, or on the viscosity coefficient

The director field dynamics

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- We assume that the driving force governing the dynamics of the director **d** is of "gradient type" $\partial_{\mathbf{d}}\mathcal{F}$, where the free-energy functional $\mathcal{F}$ is given by

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- Consequently, $\mathbf{d}$ satisfies the following equation

$$\mathbf{d}_t + \mathbf{u} \cdot \nabla_x \mathbf{d} - \mathbf{d} \cdot \nabla_x \mathbf{u} = \Delta \mathbf{d} - \partial_{\mathbf{d}} W(\mathbf{d})$$

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  where the last term accounts for stretching of the director field induced by the straining of the fluid
- The presence of the stretching term $\mathbf{d} \cdot \nabla_x \mathbf{u}$ in the **d**-equation prevents us from applying any maximum principle. Hence, we cannot find any L^∞ bound on **d** (useful in order to handle the nonlinearities)

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where p is the pressure, and

- the stress tensors are

$$\mathbb{S} = \frac{\mu(\theta)}{2} \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} \right), \quad \sigma^{nd} = -\nabla_x \mathbf{d} \odot \nabla_x \mathbf{d} + (\partial_{\mathbf{d}} W(\mathbf{d}) - \Delta \mathbf{d}) \otimes \mathbf{d}$$

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- The presence of the stretching term $\mathbf{d} \cdot \nabla_x \mathbf{u}$ in the **d**-equation prevents us from applying any maximum principle. Hence, we cannot find any L^∞ bound on **d**. We will need a weak formulation of the momentum balance

The total energy balance

$$\partial_t\left(\frac{1}{2}|\mathbf{u}|^2+e\right)+\mathbf{u}\cdot\nabla_x\left(\frac{1}{2}|\mathbf{u}|^2+e\right)+\operatorname{div}\left(p\mathbf{u}+\mathbf{q}^d+\mathbf{q}^{nd}-\mathbb{S}\mathbf{u}-\sigma^{nd}\mathbf{u}\right)$$
$$=\mathbf{g}\cdot\mathbf{u}+\operatorname{div}\left(\nabla_x\mathbf{d}\cdot(\Delta\mathbf{d}-\partial_{\mathbf{d}}W(\mathbf{d}))\right)$$

with the internal energy

$$e=\frac{|\nabla_x\mathbf{d}|^2}{2}+W(\mathbf{d})+\theta$$

and the flux

$$\mathbf{q}=\mathbf{q}^d+\mathbf{q}^{nd}=-k(\theta)\nabla_x\theta-h(\theta)(\mathbf{d}\cdot\nabla_x\theta)\mathbf{d}-\nabla_x\mathbf{d}\cdot\nabla_x\mathbf{u}\cdot\mathbf{d}$$

together with

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$$\partial_t\left(\frac{1}{2}|\mathbf{u}|^2+e\right)+\mathbf{u}\cdot\nabla_x\left(\frac{1}{2}|\mathbf{u}|^2+e\right)+\operatorname{div}\left(p\mathbf{u}+\mathbf{q}^d+\mathbf{q}^{nd}-\mathbb{S}\mathbf{u}-\sigma^{nd}\mathbf{u}\right)$$

$$=\mathbf{g}\cdot\mathbf{u}+\operatorname{div}\left(\nabla_x\mathbf{d}\cdot(\Delta\mathbf{d}-\partial_{\mathbf{d}}W(\mathbf{d}))\right)$$

with the internal energy

$$e=\frac{|\nabla_x\mathbf{d}|^2}{2}+W(\mathbf{d})+\theta$$

and the flux

$$\mathbf{q}=\mathbf{q}^d+\mathbf{q}^{nd}=-k(\theta)\nabla_x\theta-h(\theta)(\mathbf{d}\cdot\nabla_x\theta)\mathbf{d}-\nabla_x\mathbf{d}\cdot\nabla_x\mathbf{u}\cdot\mathbf{d}$$

together with

The entropy inequality

$$H(\theta)_t+\mathbf{u}\cdot\nabla_x H(\theta)+\operatorname{div}(H'(\theta)\mathbf{q}^d)$$

$$\geq H'(\theta)\left(\mathbb{S}:\nabla_x\mathbf{u}+|\Delta\mathbf{d}-\partial_{\mathbf{d}}W(\mathbf{d})|^2\right)+H''(\theta)\mathbf{q}^d\cdot\nabla_x\theta$$

holding for any smooth, non-decreasing and concave function H.

The initial and boundary conditions

In order to avoid the occurrence of boundary layers, we suppose that the boundary is impermeable and perfectly smooth imposing the complete slip boundary conditions:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \ [(\mathbb{S} + \sigma^{nd})\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

together with the no-flux boundary condition for the temperature

$$\mathbf{q}^d \cdot \mathbf{n}|_{\partial\Omega} = 0$$

and the Neumann boundary condition for the director field

$$\nabla_x d_i \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ for } i = 1, 2, 3$$

The last relation accounts for the fact that there is no contribution to the surface force from the director $\mathbf{d}$. It is also suitable for implementation of a numerical scheme.

A **weak solution** is a triple $(\mathbf{u}, \mathbf{d}, \theta)$ satisfying:

- the **momentum equations** ($\varphi \in C_0^\infty([0,T) \times \overline{\Omega}; \mathbb{R}^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$):

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$$\int_0^T \int_\Omega \Big(\mathbf{u} \cdot \partial_t \varphi + \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p \operatorname{div} \varphi \Big)$$

$$= \int_0^T \int_\Omega (\mathbb{S} + \sigma^{nd}) : \nabla_x \varphi - \int_\Omega \mathbf{g} \cdot \varphi - \int_\Omega \mathbf{u}_0 \cdot \varphi(0,\cdot)\,;$$

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- the **total energy balance** $(\varphi \in C_0^\infty([0,T) \times \overline{\Omega}),\ e_0 = \frac{\lambda}{2} |\nabla_x \mathbf{d}_0|^2 + \lambda W(\mathbf{d}_0) + \theta_0)$:

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$$\int_0^T\int_\Omega\left(\left(\frac{1}{2}|\mathbf{u}|^2+e\right)\partial_t\varphi\right)+\int_0^T\int_\Omega\left(\left(\frac{1}{2}|\mathbf{u}|^2+e\right)\mathbf{u}\cdot\nabla_x\varphi\right)$$
$$+\int_0^T\int_\Omega\Big(p\mathbf{u}+\mathbf{q}-\mathbb{S}\mathbf{u}-\sigma^{nd}\mathbf{u}\Big)\cdot\nabla_x\varphi$$
$$=\int_0^T\int_\Omega\Big(\nabla_x\mathbf{d}\cdot(\Delta\mathbf{d}-\partial_{\mathbf{d}}W(\mathbf{d}))\Big)\cdot\nabla_x\varphi-\int_0^T\int_\Omega\mathbf{g}\cdot\mathbf{u}\varphi-\int_\Omega\left(\frac{1}{2}|\mathbf{u}_0|^2+e_0\right)\varphi(0,\cdot)\,;$$

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$$+ \int_0^T \int_\Omega \left(p\mathbf{u} + \mathbf{q} - \mathbb{S}\mathbf{u} - \sigma^{nd} \mathbf{u} \right) \cdot \nabla_x \varphi$$
$$= \int_0^T \int_\Omega \Big(\nabla_x \mathbf{d} \cdot (\Delta \mathbf{d} - \partial_{\mathbf{d}} W(\mathbf{d})) \Big) \cdot \nabla_x \varphi - \int_0^T \int_\Omega \mathbf{g} \cdot \mathbf{u} \varphi - \int_\Omega \left(\frac{1}{2} |\mathbf{u}_0|^2 + e_0 \right) \varphi(0,\cdot)\,;$$

- the **entropy production inequality** ($\varphi \in C_0^\infty([0,T) \times \overline{\Omega})$, $\varphi \geq 0$):

$$\int_0^T \int_\Omega H(\theta) \partial_t \varphi + \int_0^T \int_\Omega \Big(H(\theta) \mathbf{u} + H'(\theta) \mathbf{q}^d \Big) \cdot \nabla_x \varphi$$
$$\leq - \int_0^T \int_\Omega \Big(H'(\theta) \left(\mathbb{S} : \nabla_x \mathbf{u} + |\Delta \mathbf{d} - \partial_{\mathbf{d}} W(\mathbf{d})|^2 \right) + H''(\theta) \mathbf{q}^d \cdot \nabla_x \theta \Big) \varphi - \int_\Omega H(\theta_0) \varphi(0,\cdot)$$

  for any smooth, non-decreasing and concave function H.

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Assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain of class $C^{2+\nu}$, $\mathbf{g} \in L^2((0,T)\times\Omega;\mathbb{R}^3)$,

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- The transport coefficients μ, k, and h are continuously differentiable functions satisfying

$$0 < \underline{\mu} \leq \mu(\theta) \leq \overline{\mu}, \quad 0 < \underline{k} \leq k(\theta),\ h(\theta) \leq \overline{k} \ \text{ for all } \theta \geq 0$$

and the initial data satisfy

$$\mathbf{u}_0 \in L^2(\Omega;\mathbb{R}^3),\ \operatorname{div}\mathbf{u}_0 = 0,\ \mathbf{d}_0 \in W^{1,2}(\Omega;\mathbb{R}^3),\ W(\mathbf{d}_0) \in L^1(\Omega),$$

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Then **our problem possesses a weak solution (u, d, θ)** belonging to the class

$$\mathbf{u} \in L^\infty(0,T; L^2(\Omega; \mathbb{R}^3)) \cap L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^3)),$$
$$\mathbf{d} \in L^\infty(0,T; W^{1,2}(\Omega; \mathbb{R}^3)) \cap L^2(0,T; W^{2,2}(\Omega; \mathbb{R}^3)),$$
$$W(\mathbf{d}) \in L^\infty(0,T; L^1(\Omega)) \cap L^{5/3}((0,T) \times \Omega),$$
$$\theta \in L^\infty(0,T; L^1(\Omega)) \cap L^p(0,T; W^{1,p}(\Omega)),\ 1 \leq p < 5/4,\ \theta > 0 \text{ a.e. in } (0,T) \times \Omega,$$

with the pressure p

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- It can be shown that **the solution set of our problem is weakly stable (compact) with respect to these bounds**, namely, any sequence of (weak) solutions that complies with the uniform bounds established above has a subsequence that converges to some limit
- Hence, we construct a suitable family of **approximate problems (via Faedo-Galerkin scheme + regularizing terms in the momentum equation)** whose solutions weakly converge (up to subsequences) to limit functions which solve the problem in the weak sense

Model 2: the $\mathbb{Q}$-tensorial Ball-Majumdar model

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$$\mathcal{F} = \frac{1}{2}|\nabla\mathbb{Q}|^2 + f_B(\theta, \mathbb{Q}) - \theta \log \theta$$

where f_B is bulk the configuration potential:

- $f_B(\theta, \mathbb{Q}) = f(\mathbb{Q}) - U(\theta)G(\mathbb{Q})$
- f is the convex l.s.c. and singular Ball-Majumdar potential
- U changes in sign at a critical temperature: $U(\theta) = \alpha(\theta - \theta^*)$ for $\theta \sim \theta^*$ with a controlled growth for large θ
- e.g. $G(\mathbb{Q}) = \mathrm{tr}(\mathbb{Q}^2)$

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Theorem [E. Feireisl, E.R., G. Schimperna, A. Zarnescu, paper in preparation] There exists *at least* one weak solution to a system coupling

- a *weak momentum equation* for $\mathbf{u}$
- a *gradient-type equation for* $\mathbb{Q}$
- an *entropy inequality+total energy balance* for θ

for finite-energy initial data.

The generalized principle of virtual powers: damage phenomena

The generalized principle of virtual powers in damage phenomena

The scope: The analysis of the initial boundary-value problem for the following PDE system:

$$
\begin{aligned}
&c(\theta)\theta_t + \chi_t\theta - \mathrm{div}(k(\theta)\nabla\theta) = g \\
&\mathbf{u}_{tt} - \mathrm{div}(\chi\varepsilon(\mathbf{u}_t) + \chi\varepsilon(\mathbf{u})) = \mathbf{f} \\
&\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + W'(\chi) \ni -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta
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- θ is the absolute temperature of the system
- $\mathbf{u}$ the vector of *small displacements*
- χ is the damage parameter, assessing the soundness of the material in *damage* (for the completely *damaged* $\chi = 0$ and the *undamaged* state $\chi = 1$, respectively, while $0 < \chi < 1$: *partial damage*)

[joint works with R. Rossi, J. Differential Equations and Appl. Math (2008) and preprint arXiv:1205.3578v1 (2012)]

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$\Longrightarrow$ We shall approximate the system with a non-degenerating one, where we replace the momentum equation with

$$\mathbf{u}_{tt} - \operatorname{div}((\chi + \delta)\varepsilon(\mathbf{u}_t) + (\chi + \delta)\varepsilon(\mathbf{u})) = \mathbf{f} \qquad \text{for } \delta > 0$$

It seems to us that *both* the coefficients need to be truncated when taking the degenerate limit in the momentum equation. Indeed, on the one hand the truncation in front of $\varepsilon(\mathbf{u}_t)$ allows us to deal with the *main part* of the elliptic operator. On the other hand, in order to pass to the limit in the quadratic term on the right-hand side of χ-eq., we will also need to truncate the coefficient of $\varepsilon(\mathbf{u})$.

Free energy and Dissipation, cf. [M. Frémond, Phase Change in Mechanics, Lecture Notes of the Unione Matematica Italiana 13, Springer-Verlag, Berlin, 2012]

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The free-energy $\mathcal{F}$:

$$\mathcal{F} = \int_\Omega \Big(f(\theta) + \chi \frac{|\varepsilon(\mathbf{u})|^2}{2} + \frac{a_s(\chi,\chi)}{2} + W(\chi) - \theta\chi \Big)\, \mathrm{d}x$$

- f is a concave function
- $a_s(z_1, z_2) := \int_\Omega \int_\Omega \frac{(\nabla z_1(x) - \nabla z_1(y)) \cdot (\nabla z_2(x) - \nabla z_2(y))}{|x-y|^{d+2(s-1)}}\, \mathrm{d}x\, \mathrm{d}y$ is the bilinear form associated to the fractional $s-$Laplacian A_s
- $s > d/2$: we need the embedding of $H^s(\Omega)$ into $C^0(\overline{\Omega})$
- $W = \widehat{\beta} + \widehat{\gamma}$, $\widehat{\gamma} \in C^2(\mathbb{R})$, $\widehat{\beta}$ proper, convex, l.s.c., $\overline{\mathrm{dom}(\widehat{\beta})} = [0,1]$

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The pseudo-potential $\mathcal{P}$:

$$\mathcal{P} = \frac{k(\theta)}{2} |\nabla\theta|^2 + \frac{1}{2}|\chi_t|^2 + \chi \frac{|\varepsilon(\mathbf{u}_t)|^2}{2} + I_{(-\infty,0]}(\chi_t)$$

- k the heat conductivity: coupled conditions with the specific heat $c(\theta) = f(\theta) - \theta f'(\theta)$
- $I_{(-\infty,0]}(\chi_t) = 0$ if $\chi_t \in (-\infty, 0]$, $I_{(-\infty,0]}(\chi_t) = +\infty$ otherwise

The modelling

The momentum equation

$$\mathbf{u}_{tt} - \operatorname{div} \sigma = \mathbf{f} \quad \left(\sigma = \sigma^{nd} + \sigma^{d} = \frac{\partial \mathcal{F}}{\partial \varepsilon(\mathbf{u})} + \frac{\partial \mathcal{P}}{\partial \varepsilon(\mathbf{u}_t)} \right) \quad \text{becomes}$$

$$\boxed{\mathbf{u}_{tt} - \operatorname{div}(\chi \varepsilon(\mathbf{u}_t) + \chi \varepsilon(\mathbf{u})) = \mathbf{f}}$$

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The principle of virtual powers

$$B - \operatorname{div} \mathbf{H} = 0 \quad \left(B = \frac{\partial \mathcal{F}}{\partial \chi} + \frac{\partial \mathcal{P}}{\partial \chi_t}, \mathbf{H} = \frac{\partial \mathcal{F}}{\partial \nabla \chi}\right) \quad \text{becomes}$$

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The internal energy balance

$$e_t + \operatorname{div}\mathbf{q} = g + \sigma : \varepsilon(\mathbf{u}_t) + B\chi_t + \mathbf{H}\cdot\nabla\chi_t \quad \left(e = \mathcal{F} - \theta\frac{\partial \mathcal{F}}{\partial \theta}, \quad \mathbf{q} = \frac{\partial \mathcal{P}}{\partial \nabla\theta}\right)$$

becomes

$$\boxed{c(\theta)\theta_t + \chi_t\theta - \operatorname{div}(k(\theta)\nabla\theta) = g + |\chi_t|^2 + \chi|\varepsilon(\mathbf{u}_t)|^2}$$

Main difficulties and weak formulation

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- We replace the momentum equation with a non-degenerating one

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- We have to handle the *nonlinear coupling* between the single equations: in the heat equation (even with the *small perturbation assumption*)

$$\mathrm{c}(\theta)\theta_t + \chi_t\theta - \mathrm{div}(k(\theta)\nabla\theta) = g$$

and in the phase equation

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + W'(\chi) \ni -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta \tag{2}$$

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$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + W'(\chi) \ni -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta \tag{2}$$

- A major difficulty stems from the simultaneous presence in (2) of $\partial I_{(-\infty,0]}(\chi_t)$ and $W'(\chi)$ and from the low regularities of $-\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta$ on the r.h.s. $\Longrightarrow$ follow the approach of [Heinemann, Kraus, WIAS preprints (2010)] and consider a suitable weak formulation of (2) consisting of a one-sided variational inequality + an energy inequality $\Longrightarrow$ **generalized principle of virtual powers**

Main difficulties and weak formulation

- We replace the momentum equation with a non-degenerating one

$$\mathbf{u}_{tt} - \operatorname{div}((\chi + \delta)\varepsilon(\mathbf{u}_t) + (\chi + \delta)\varepsilon(\mathbf{u})) = \mathbf{f}, \quad \delta > 0 \tag{1}$$

- We have to handle the *nonlinear coupling* between the single equations: in the heat equation (even with the *small perturbation assumption*)

$$c(\theta)\theta_t + \chi_t\theta - \operatorname{div}(k(\theta)\nabla\theta) = g$$

  and in the phase equation

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + W'(\chi) \ni -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta \tag{2}$$

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- For the analysis of the degenerate limit $\delta \searrow 0$ we have carefully adapted techniques from [Bouchitté, Mielke, Roubíček, ZAMP (2009)] and [Mielke, Roubíček, Zeman, Comput. Methods Appl. Mech. Engrg. (2011)] to the case of a *rate-dependent* equation for χ, also coupled with the temperature equation.

Energy vs Enthalpy

In order to deal with the low regularity of θ, rewrite the internal energy equation

$$\mathrm{c}(\theta)\theta_t + \chi_t\theta - \mathrm{div}(k(\theta)\nabla\theta) = g$$

as the enthalpy equation

$$w_t + \chi_t\Theta(w) - \mathrm{div}(K(w)\nabla w) = g \quad \text{where}$$

$$w = h(\theta) := \int_0^\theta \mathrm{c}(s)\,\mathrm{d}s, \quad \Theta(w) := \begin{cases} h^{-1}(w) & \text{if } w \geq 0, \\ 0 & \text{if } w < 0, \end{cases} \quad K(w) := \frac{k(\Theta(w))}{\mathrm{c}(\Theta(w))}$$

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We assume that

- $\mathrm{c} \in C^0([0,+\infty);[0,+\infty))$
- $\exists\, \sigma_1 \geq \sigma > \frac{2d}{d+2}$: $\quad c_0(1+\theta)^{\sigma-1} \leq \mathrm{c}(\theta) \leq c_1(1+\theta)^{\sigma_1-1} \Longrightarrow h$ is strictly increasing
- the function $k : [0,+\infty) \to [0,+\infty)$ is continuous, and

$$\exists c_2,\, c_3 > 0 \;\; \forall\theta \in [0,+\infty) : \quad c_2\mathrm{c}(\theta) \leq k(\theta) \leq c_3(\mathrm{c}(\theta)+1)$$

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$$\exists c_2,\ c_3 > 0 \ \ \forall\theta \in [0,+\infty) : \quad c_2 c(\theta) \leq k(\theta) \leq c_3(c(\theta)+1)$$

$\Longrightarrow \exists\,\bar{c} > 0 \ \ \forall\, w \in \mathbb{R} :\ c_2 \leq K(w) \leq \bar{c}$

$\Longrightarrow$ for every $s \in (1,\infty)\ \exists\, C_s > 0 \ \ \forall\, w \in L^1(\Omega) : \quad \|\Theta(w)\|_{L^s(\Omega)} \leq C_s(\|w\|_{L^{s/\sigma}(\Omega)}^{1/\sigma} + 1)$

The approximating non-degenerate Problem $[P_\delta]$

Given $\delta > 0$, take $W' = \partial I_{[0,+\infty)} + \gamma$, $\gamma \in C^1(\mathbb{R})$, find (measurable) functions

$$w \in L^r(0,T;W^{1,r}(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)$$
$$\mathbf{u} \in H^1(0,T;H^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H_0^1(\Omega)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))$$
$$\chi \in L^\infty(0,T;H^s(\Omega)) \cap H^1(0,T;L^2(\Omega))$$

for every $1 \le r < \frac{d+2}{d+1}$, fulfilling the initial conditions

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \mathbf{u}_t(0,x) = \mathbf{v}_0(x) \qquad \text{for a.e. } x \in \Omega$$
$$\chi(0,x) = \chi_0(x) \qquad \text{for a.e. } x \in \Omega$$

the equations (for every $\varphi \in \mathrm{C}^0([0,T];W^{1,r'}(\Omega)) \cap W^{1,r'}(0,T;L^{r'}(\Omega))$ and $t \in (0,T]$)

$$\int_\Omega \varphi(t)\,w(t)(\mathrm{d}x) - \int_0^t\int_\Omega w\varphi_t\,\mathrm{d}x + \int_0^t\int_\Omega \chi_t\Theta(w)\varphi\,\mathrm{d}x + \int_0^t\int_\Omega K(w)\nabla w\nabla\varphi\,\mathrm{d}x$$
$$= \int_0^t\int_\Omega g\varphi + \int_\Omega w_0\varphi(0)\,\mathrm{d}x$$
$$\mathbf{u}_{tt} - \mathrm{div}\left((\chi+\delta)\varepsilon(\mathbf{u}_t) + (\chi+\delta)\varepsilon(\mathbf{u})\right) = \mathbf{f} \text{ in } H^{-1}(\Omega;\mathbb{R}^d) \text{ a.e. in } (0,T)$$

and the subdifferential inclusion "in a suitable sense"

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + \partial I_{[0,+\infty)}(\chi) + \gamma(\chi) \ni -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \quad \text{in } H^{-s}(\Omega) \text{ and a.e. in } (0,T)$$

Generalized principle of virtual powers for $\delta > 0$ [E.R., R. Rossi, preprint arXiv:1205.3578v1 (2012)]

[Theorem 1] ($\delta > 0$) Under the previous assumptions on the data, then,

[1.] Problem [P_δ] admits a *weak solution* $(w, \mathbf{u}, \chi)$, which, beside fulfilling the enthalpy and momentum equations, satisfies $\chi_t(x,t) \le 0$ for almost all $t \in (0,T)$, and $(\forall \varphi \in L^2(0,T; H^s_+(\Omega)) \cap L^\infty(Q))$ the *one-sided* inequality

$$\int_0^T \int_\Omega \chi_t \varphi + a_s(\chi, \varphi) + \xi\varphi + \gamma(\chi)\varphi + \frac{|\varepsilon(\mathbf{u})|^2}{2}\varphi - \Theta(w)\varphi \le 0$$

with $\xi \in \partial I_{[0,+\infty)}(\chi)$ in the following sense:

$$\xi \in L^1(0,T; L^1(\Omega)), \ \langle \xi(t), \varphi - \chi(t) \rangle_{H^s(\Omega)} \le 0 \quad \forall\, \varphi \in H^s_+(\Omega), \text{ a.e. } t \in (0,T)$$

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and the energy inequality for all $t \in (0,T]$, for $s = 0$, and for almost all $0 < s \le t$:

$$\int_s^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \frac{1}{2} a_s(\chi(t), \chi(t)) + \int_\Omega W(\chi(t))\,\mathrm{d}x$$
$$\le \frac{1}{2} a_s(\chi(s), \chi(s)) + \int_\Omega W(\chi(s))\,\mathrm{d}x + \int_s^t \int_\Omega \chi_t \left(-\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \right) \mathrm{d}x\,\mathrm{d}r$$

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$$\begin{aligned} &\int_s^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \frac{1}{2} a_s(\chi(t), \chi(t)) + \int_\Omega W(\chi(t))\,\mathrm{d}x \\ &\le \frac{1}{2} a_s(\chi(s), \chi(s)) + \int_\Omega W(\chi(s))\,\mathrm{d}x + \int_s^t \int_\Omega \chi_t \left(-\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \right) \mathrm{d}x\,\mathrm{d}r \end{aligned}$$

[2.] Suppose in addition that $g(x,t) \ge 0$, $\theta_0 > \underline{\theta}_0 \ge 0$ a.e. Then $\theta(x,t) := \Theta(w(x,t)) \ge \underline{\theta}_0 \ge 0$ a.e.

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Uniqueness of solutions for the irreversible system, even in the isothermal case, is still an open problem. This is mainly due to the doubly nonlinear character of the χ equation.

Generalized principle of virtual powers vs classical phase inclusion

Generalized principle of virtual powers vs classical phase inclusion

- Any *weak solution* $(w, \mathbf{u}, \chi)$ fulfills the total energy inequality for all $t \in (0, T]$, for $s = 0$, and for almost all $0 < s \le t$

$$
\begin{aligned}
&\int_\Omega w(t)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\mathbf{u}_t(t)|^2\,\mathrm{d}x + \int_s^t \int_\Omega |\chi_t|^2\,\mathrm{d}x + \int_s^t (\chi+\delta)|\varepsilon(\mathbf{u}_t)|^2 \\
&\quad + \frac{1}{2}(\chi(t)+\delta)|\varepsilon(\mathbf{u}(t))|^2 + \frac{1}{2}a_s(\chi(t),\chi(t)) + \int_\Omega W(\chi(t))\,\mathrm{d}x \\
&\le \int_\Omega w(s)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\mathbf{u}_t(s)|^2\,\mathrm{d}x + \frac{1}{2}(\chi(s)+\delta)|\varepsilon(\mathbf{u}(s))|^2 + \frac{1}{2}a_s(\chi(s),\chi(s)) \\
&\quad + \int_\Omega W(\chi(s))\,\mathrm{d}x + \int_s^t \int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x + \int_s^t \int_\Omega g\,\mathrm{d}x
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- If $(w, \mathbf{u}, \chi)$ are "more regular" and satisfy the notion of *weak solution*, then, differentiating the energy inequality and using the chain rule, we conclude that $(w, \mathbf{u}, \chi, \xi)$ comply with

$$\left\langle \chi_t(t) + A_s(\chi(t)) + \xi(t) + \gamma(\chi(t)) + \frac{|\varepsilon(\mathbf{u})|^2}{2} - \Theta(w(t)), \chi_t(t)\right\rangle_{H^s(\Omega)} \le 0 \text{ for a.e.} t$$

Using the *one-sided* inequality we obtain the classical phase inclusion:

$$\exists\,\zeta \in L^2(0,T;L^2(\Omega)) \text{ with } \zeta(x,t) \in \partial I_{(-\infty,0]}(\chi_t(x,t)) \text{ a.e. s.t.}$$

$$\chi_t + \zeta + A_s\chi + \xi + \gamma(\chi) = -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \text{ a.e.}$$

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- The presence of the s-Laplacian with $s > d/2 \Longrightarrow$ an estimate for χ in $L^\infty(0,T;H^s(\Omega))$ (from the total energy balance) $\Longrightarrow$ a suitable regularity estimate on the displacement variable $\mathbf{u} \Longrightarrow$ an $L^\infty(0,T;L^2(\Omega))$-bound on the quadratic nonlinearity $|\varepsilon(\mathbf{u})|^2$ on the right-hand side of

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- We pass to the limit in a carefully designed time-discretization scheme
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- A Boccardo-Gallouët-type estimates combined with the Gagliardo-Nirenberg inequality applied to the enthalpy equation in order to obtain an $L^r(0,T;W^{1,r}(\Omega))$-estimate on the enthalpy w (and hence on $\Theta(w)$)

$$w_t + \chi_t\Theta(w) - \mathrm{div}(K(w)\nabla w) = g$$

The total energy inequality in the degenerating case $\delta \searrow 0$

Rewrite the momentum equation

$$\partial_t^2 \mathbf{u}_\delta - \mathrm{div}((\chi + \delta)\varepsilon(\partial_t \mathbf{u}_\delta)) - \mathrm{div}((\chi + \delta)\varepsilon(\mathbf{u}_\delta)) = \mathbf{f}$$

using the new variables (*quasi-stresses*) $\boldsymbol{\mu}_\delta := \sqrt{\chi_\delta + \delta}\, \varepsilon(\partial_t \mathbf{u}_\delta)$, and $\boldsymbol{\eta}_\delta := \sqrt{\chi_\delta + \delta}\, \varepsilon(\mathbf{u}_\delta)$:

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$$\partial_t^2 \mathbf{u}_\delta - \mathrm{div}(\sqrt{\chi + \delta}\,\boldsymbol{\mu}_\delta) - \mathrm{div}(\sqrt{\chi + \delta}\,\boldsymbol{\eta}_\delta) = \mathbf{f}$$

The total energy inequality for $(w_\delta, \mathbf{u}_\delta, \chi_\delta)$ is

$$\begin{aligned}
&\int_\Omega w_\delta(t)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\partial_t \mathbf{u}_\delta(t)|^2 \,\mathrm{d}x + \int_s^t \int_\Omega |\partial_t \chi_\delta|^2 \,\mathrm{d}x + \frac{1}{2}\int_s^t |\boldsymbol{\mu}_\delta(r)|^2 \\
&\quad + \frac{|\boldsymbol{\eta}_\delta(t)|^2}{2} + \frac{1}{2} a_s(\chi_\delta(t), \chi_\delta(t)) + \int_\Omega W(\chi_\delta(t))\,\mathrm{d}x \\
&\leq \int_\Omega w_\delta(s)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\partial_t \mathbf{u}_\delta(s)|^2 \,\mathrm{d}x + \frac{|\boldsymbol{\eta}_\delta(s)|^2}{2} + \frac{1}{2} a_s(\chi_\delta(s), \chi_\delta(s)) \\
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The degenerate problem ($\delta = 0$): the existence theorem [E.R., R. Rossi, preprint arXiv:1205.3578v1 (2012)]

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[Theorem 2] **($\delta = 0$)** Under the previous assumptions, there exist

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such that it holds true (a.e. in any open set $A \subset \Omega \times (0,T)$: $\chi > 0$ a.e. in A)

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the weak enthalpy equation and the *weak* momentum and phase relations

$$\partial_t^2 \mathbf{u} - \mathrm{div}(\sqrt{\chi}\,\boldsymbol{\mu}) - \mathrm{div}(\sqrt{\chi}\,\boldsymbol{\eta})) = \mathbf{f} \quad \text{in } H^{-1}(\Omega;\mathbb{R}^d), \text{ a.e. in } (0,T),$$

$$\int_0^T \int_\Omega (\partial_t \chi + \gamma(\chi))\,\varphi\,\mathrm{d}x + \int_0^T a_s(\chi,\varphi) \leq \int_0^T \int_\Omega \left(-\frac{1}{2\chi}|\boldsymbol{\eta}|^2 + \Theta(w)\right)\varphi\,\mathrm{d}x$$

for all $\varphi \in L^2(0,T;H^s_+(\Omega)) \cap L^\infty(Q)$ with $\mathrm{supp}(\varphi) \subset \{\chi > 0\}$,

The degenerate problem ($\delta = 0$): the existence theorem [E.R., R. Rossi, preprint arXiv:1205.3578v1 (2012)]

[Theorem 2] ($\delta = 0$) Under the previous assumptions, there exist

$$\mathbf{u} \in W^{1,\infty}(0,T;L^2(\Omega)) \cap H^2(0,T;H^{-1}(\Omega)),\ \boldsymbol{\mu} \in L^2(0,T;L^2(\Omega)),\ \boldsymbol{\eta} \in L^\infty(0,T;L^2(\Omega)),$$

$$w \in L^r(0,T;W^{1,r}(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)$$

$$\chi \in L^\infty(0,T;H^s(\Omega)) \cap H^1(0,T;L^2(\Omega)), \quad \chi(x,t) \geq 0, \quad \chi_t(x,t) \leq 0 \text{ a.e.}$$

such that it holds true (a.e. in any open set $A \subset \Omega \times (0,T)$: $\chi > 0$ a.e. in A)

$$\boldsymbol{\mu} = \sqrt{\chi}\,\varepsilon(\mathbf{u}_t),\ \boldsymbol{\eta} = \sqrt{\chi}\,\varepsilon(\mathbf{u}),$$

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for all $\varphi \in L^2(0,T;H^s_+(\Omega)) \cap L^\infty(Q)$ with $\mathrm{supp}(\varphi) \subset \{\chi > 0\}$,

together with the **total energy inequality** (for almost all $t \in (0,T]$)

$$\int_\Omega w(t)(\mathrm{d}x) + \int_0^t\int_\Omega |\chi_t|^2\,\mathrm{d}x + \frac{1}{2}\int_0^t |\boldsymbol{\mu}(r)|^2 + \int_\Omega W(\chi(t))\,\mathrm{d}x + \mathcal{J}(t) = \int_\Omega w_0\,\mathrm{d}x$$

$$+\frac{1}{2}\int_\Omega |\mathbf{v}_0|^2\,\mathrm{d}x + \frac{1}{2}\chi_0|\varepsilon(\mathbf{u}_0)|^2 + \frac{1}{2}a_s(\chi_0,\chi_0) + \int_\Omega W(\chi_0)\,\mathrm{d}x + \int_0^t\int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x\mathrm{d}r + \int_0^t\int_\Omega g\,\mathrm{d}x$$

$$\text{with } \int_0^t \mathcal{J}(r)\,\mathrm{d}r \geq \frac{1}{2}\int_0^t \left(\int_\Omega |\mathbf{u}_t(r)|^2\,\mathrm{d}x + |\boldsymbol{\eta}(r)|^2 + a_s(\chi(r),\chi(r))\right)$$

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coincides with

$$\int_0^T \int_\Omega \chi_t \varphi + a_s(\chi, \varphi) + \xi\varphi + \gamma(\chi)\varphi + \frac{|\varepsilon(\mathbf{u})|^2}{2}\varphi - \Theta(w)\varphi \leq 0$$

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$\forall \varphi \in L^2(0, T; H^s_+(\Omega)) \cap L^\infty(Q)$ and with $\xi \in \partial I_{[0,+\infty)}(\chi)$. Subtracting from the degenerate total energy inequality the weak enthalpy equation tested by 1, we recover (a.e. in $(0, T]$) the energy inequality:

$$\int_0^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \frac{1}{2} a_s(\chi(t), \chi(t)) + \int_\Omega W(\chi(t))\,\mathrm{d}x$$
$$\leq \frac{1}{2} a_s(\chi_0, \chi_0) + \int_\Omega W(\chi_0)\,\mathrm{d}x + \int_0^t \int_\Omega \chi_t \left(-\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \right) \mathrm{d}x\,\mathrm{d}r$$

Work in progress: an entropic formulation for the damage phenomena

We worked here with the small perturbation assumption, i.e. neglecting the quadratic contribution on the r.h.s in the internal energy balance:

$$\theta_t + \chi_t\theta - \Delta\theta = |\chi_t|^2 + \chi|\varepsilon(\mathbf{u}_t)|^2$$

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✓ the entropy production

$$\int_0^T \int_\Omega \Big((\log\theta + \chi)\, \partial_t\varphi - \nabla \log\theta \cdot \nabla\varphi \Big) dxdt$$
$$\leq \int_0^T \int_\Omega \frac{1}{\theta} \left(-|\chi_t|^2 - \chi|\varepsilon(\mathbf{u}_t)|^2 - \nabla \log\theta \cdot \nabla\theta \right) \varphi dxdt$$

for every test function $\varphi \in \mathcal{D}(\overline{Q}_T)$, $\varphi \geq 0$

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  for every test function $\varphi\in\mathcal{D}(\overline{Q}_T)$, $\varphi\ge 0$ and

- ✓ the energy conservation

$$E(t) = E(0) \text{ for a.e. } t\in[0,T],$$

  where

$$E \equiv \int_\Omega\left(\theta + W(\chi) + \frac{1}{2}a_s(\chi,\chi) + \frac{|\mathbf{u}_t|^2}{2} + \chi\frac{|\varepsilon(\mathbf{u})|^2}{2}\right)dx\,.$$

This is still a work in progress (with R. Rossi)...

Possible further application

- A fluid-mechanical theory for **two-phase mixtures of fluids** faces a well known mathematical difficulty:
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$$\partial_t \theta + \operatorname{div}(\theta \mathbf{v}) + \operatorname{div} \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{v} + |\nabla_x \mu|^2 \tag{2}$$

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Entropic notion of solution is needed in order to interpret the internal energy balance (2) ...

Thanks for your attention!

cf. http://www.mat.unimi.it/users/rocca/