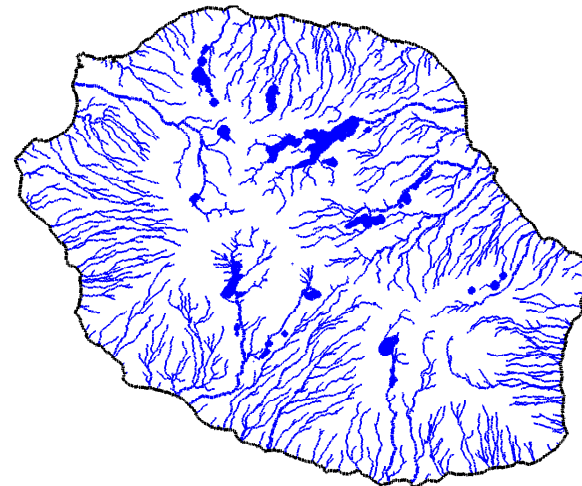


Lakes and Rivers in the Landscape: A Quasi-Variational Inequality Approach

John W. Barrett¹ and Leonid Prigozhin²

¹Imperial College, London, UK

²Ben-Gurion University of the Negev, Israel

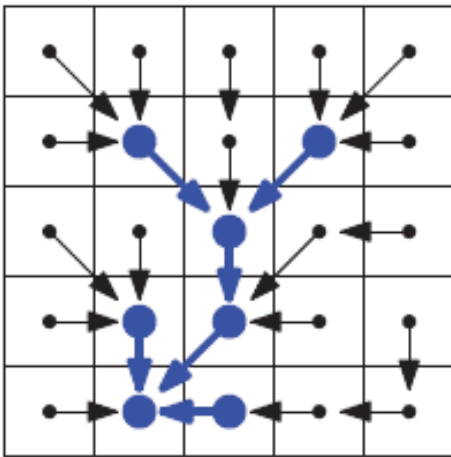


DEM data and D8 algorithm

Usually, Geographic Information Systems (GIS) employ a modification of the basic **D8** (deterministic eight directions) lattice algorithm to automatically extract a drainage network from the Digital Elevation Model (**DEM**) datasets.

DEMs contain elevations of the earth surface in a regular grid of cells.

Each cell gets the same precipitation; the flow from each cell is directed into that of its neighbors towards which the surface decreases faster.



River networks are plotted as the sets of cells whose accumulating water flux exceeds some threshold.

The set of cells contributing their water to a river is the river basin; watersheds are basin boundaries.

D8

A problem: Flow direction is not determined in pits. Usually, the pits are filled but this leads to appearance of flat areas for which flow routing is also problematic.

Flat areas are abundant also for low-vertical-resolution DEMs of reliefs without well-developed topographic features.

Solutions: Various heuristic iterative algorithms for drainage enforcement in flat areas (directing all flow to the outlet, breaching outlets, creating artificial gradients, etc.)

Despite the difficulties of flow routing in low relief areas, modifications of D8 have been implemented as very efficient routines in various GIS and, usually, produce useful and realistic drainage networks.

We use a different approach: an approximation of a continuous model, the zero-repose-angle limit of a growing sandpile model.

Growing sandpile model [LP86,96]

$$\partial_t w + \nabla \cdot \mathbf{q} = f \quad (x \in \Omega \subset \mathbb{R}^2, t > 0)$$

$$w|_{t=0} = w_0 \in W_0^1(\Omega), \quad w|_{\partial\Omega} = 0$$

where

$y = w(x, t)$ – pile surface

$f(x, t) \geq 0$ – source intensity

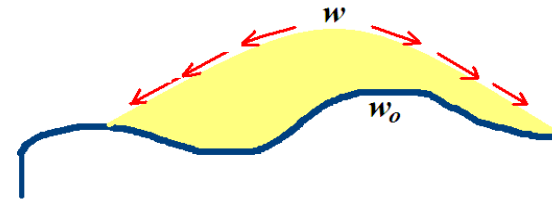
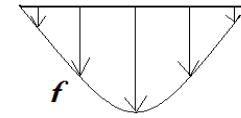
$w_0(x)$ – support surface,

$\mathbf{q}(x, t)$ – horizontal projection of sand surface flux

Sand is characterized by $k_0 = \tan \alpha_r$.

The constitutive relation:

- 1) \mathbf{q} is towards the steepest descent of w ;
- 2) $|\nabla w| < k_0 \Rightarrow \mathbf{q} = 0$;
- 3) $w > w_0 \Rightarrow |\nabla w| \leq k_0$,



More convenient are variational formulations.

Growing sandpile model

The constitutive relation

- 1) \mathbf{q} is towards the steepest descent of w ;
- 2) $|\nabla w| < k_0 \Rightarrow \mathbf{q} = 0$;
- 3) $w > w_0 \Rightarrow |\nabla w| \leq k_0$,

yields an equilibrium condition
supplemented by

$$\begin{aligned} |\nabla w| &\leq M(w) \\ -\nabla w \cdot \mathbf{q} &= M(w)|\mathbf{q}|. \end{aligned}$$

Here

$$M(w)(x, t) = \begin{cases} k_0 & w(x, t) > w_0(x), \\ \max(k_0, |\nabla w_0(x)|) & \text{otherwise,} \end{cases}$$

changes discontinuously at $w = w_0$ if $|\nabla w_0| > k_0$.

Flux \mathbf{q} can be excluded if only the evolving free surface w is of interest.

Primal QVI

Excluding the surface flux we obtain an evolutionary QVI:

$$\begin{aligned} &\text{Find } w(x, t) \text{ such that for a.a. } t \\ &w \in K(w) \text{ and } (\partial_t w - f, \varphi - w) \geq 0 \quad \forall \varphi \in K(w), \\ &w|_{t=0} = w_0, \end{aligned}$$

where, for any $\eta \in C(\overline{\Omega})$, the closed convex set

$$K(\eta) := \left\{ \varphi \in W_0^{1,\infty}(\Omega) : |\nabla \varphi| \leq M(\eta) \text{ a.e. in } \Omega \right\}.$$

If $|\nabla w_0| \leq k_0$ a.e. in Ω , the inequality is variational; this case is well studied [LP96; Aronson, Evans, Wu 96]. Dual formulations in terms of flux have been also proposed for the VI case [Barrett and LP 06; Dumont and Igbida 09].

For the QVI case we will use a mixed formulation.

Mixed formulation

The inequality $|\nabla w| \leq M(w)$ yields $M(w)|\boldsymbol{\psi}| + \nabla w \cdot \boldsymbol{\psi} \geq 0$ a.e. for any test field $\boldsymbol{\psi}$.

Noting that $M(w)|\mathbf{q}| + \nabla w \cdot \mathbf{q} = 0$ we formally obtain

$$(M(w), |\boldsymbol{\psi}| - |\mathbf{q}|) - (w, \nabla \cdot \{\boldsymbol{\psi} - \mathbf{q}\}) \geq 0, \quad \forall \boldsymbol{\psi}.$$

Supplemented by the balance equation

$$\partial_t w + \nabla \cdot \mathbf{q} = f, \quad w|_{t=0} = w_0, \quad w|_{\partial\Omega} = 0$$

this provides for a mixed variational formulation written for both variables.

Discontinuity of $M(w)$ makes the problem especially difficult.

We approximate M by $M_\varepsilon : C(\overline{\Omega}) \rightarrow C(\overline{\Omega})$.

Regularized problem

For a small $\varepsilon > 0$ we approximate w_0 by $w_{0,\varepsilon} \in C^1 \cap W_0^1$ and replace the jump of M at $w = w_0$ by a continuous transition:

$$M_\varepsilon(w) = \begin{cases} k_0 & w \geq w_{0,\varepsilon} + \varepsilon, \\ k_1 + (k_0 - k_1) \frac{w - w_{0,\varepsilon}}{\varepsilon} & w \in (w_{0,\varepsilon}, w_{0,\varepsilon} + \varepsilon), \\ k_1 := \max(k_0, |\nabla w_{0,\varepsilon}|) & w \leq w_{0,\varepsilon}. \end{cases}$$

It has been proved [Barrett and LP, 12] there exists a weak solution $\{q, w\}$ to the regularized mixed problem with w being a weak solution to the regularized primal QVI.

Existence of a solution to the regularized primal QVI follows also from a general result in [Rodrigues and Santos, 12].

The lake and river formation model is obtained by sending k_0 to zero. Both w and q need to be found.

A different approach: see [Dorfman and Evans 09].

Lakes and rivers: the $k_0 \rightarrow 0$ limit

Let now f be the precipitation rate; we assume rain water neither penetrates the soil nor evaporates, just flows towards the steepest descent and accumulates into lakes at local depressions of the earth relief.

As the lake overflows, it passes additional water along a 1d river, possibly, to another lake below. The steepest descent lines also can merge. **Singular surface fluxes can be expected.**

The model [LP, 94] is similar to that for sand but, for $k_0 = 0$, the piles become lakes:

$$w > w_0 \Rightarrow \nabla w = \mathbf{0}.$$

In the $k_0 \rightarrow 0$ limit the flow in lakes becomes undetermined in our model [LP, 94]. However, this flow influences neither the free boundary w nor the water flux \mathbf{q} in the hills, where $w = w_0$.

Complications...

- Small changes of the relief may in some cases lead to a significant reconstruction of the river network.
In practice, DEM resolution should be sufficient to make river valleys discernible.
- We have already regularized the problem by smoothing the operator M .
Now we also replace $k_0 = 0$ by a small positive k_0 to avoid the non-uniqueness of q in the lakes.
- For $k_0 > 0$ convergence was shown [Barrett and LP, 12] for an approximation of the regularized mixed problem based on Raviart-Thomas elements for q and p.-w. constant for w . This approximation of flux “smears” rivers; it is inconvenient for hydrological applications.
Here we explore a different approximation.

Approximation of rivers

We approximate q by a network of fluxes along the edges of a triangulation.

Let \mathcal{T}^h be a regular partitioning of Ω^h , a polygonal approximation of Ω , into triangles σ ; $h = \max_{\sigma}(\text{diam}\{\sigma\})$. By \mathcal{N}^h and \mathcal{E}^h we denote the sets of vertices (nodes) and oriented edges of \mathcal{T}^h , respectively.

The edge $e_{kl} \in \mathcal{E}^h$ is oriented from the node v_k to the node v_l ; the unit vector in the direction of edge e is \mathbf{i}_e .

Let $0 = t^0 < t^1 < \dots < t^n = T$ and $\tau^n = t^n - t^{n-1}$.

For every $t = t^n$ the flux q^n is sought in the set

$$V^h := \{\boldsymbol{\psi} \in [\mathcal{M}(\Omega^h)]^2 : \boldsymbol{\psi} = \sum_{e \in \mathcal{E}^h} \psi_e \mathbf{i}_e d\mathcal{H}^1(e)\}$$

where $\psi_e \in \mathbb{R}^1$, $d\mathcal{H}^1(e)$ is the one-dimensional Hausdorff measure supported on the edge e , and $\mathcal{M}(\Omega^h)$ is the Banach space of bounded Radon measures, $\mathcal{M}(\Omega^h) = [C(\overline{\Omega^h})]^*$.

Approximation of rivers

We approximate q by a network of fluxes along the edges of a triangulation.

Let \mathcal{T}^h be a regular partitioning of Ω^h , a polygonal approximation of Ω , into triangles σ ; $h = \max_{\sigma}(\text{diam}\{\sigma\})$. By \mathcal{N}^h and \mathcal{E}^h we denote the sets of vertices (nodes) and oriented edges of \mathcal{T}^h , respectively.

The edge $e_{kl} \in \mathcal{E}^h$ is oriented from the node v_k to the node v_l ; the unit vector in the direction of edge e is \mathbf{i}_e .

Let $0 = t^0 < t^1 < \dots < t^n = T$ and $\tau^n = t^n - t^{n-1}$.

For every $t = t^n$ the flux q^n is sought in the set

$$V^h := \{\boldsymbol{\psi} \in [\mathcal{M}(\Omega^h)]^2 : \boldsymbol{\psi} = \sum_{e \in \mathcal{E}^h} \psi_e \mathbf{i}_e d\mathcal{H}^1(e)\}$$

The approximate surface w^n is assumed p.w. linear:

$$w^n \in W_0^h := \{\phi \in C_0(\Omega^h) : \phi|_{\sigma} \text{ is linear } \forall \sigma \in \mathcal{T}^h\}$$

Discrete mixed formulation

Find $w^n \in W_0^h$ and $\mathbf{q}^n \in V^h$ such that

$$\left(\frac{w^n - w^{n-1}}{\tau^n} - f^n, \phi \right) - \langle \nabla \phi, \mathbf{q}^n \rangle = 0 \quad \forall \phi \in W_0^h,$$

$$\langle M_\varepsilon^h(w^n), |\boldsymbol{\psi}| - |\mathbf{q}^n| \rangle + \langle \nabla w^n, \boldsymbol{\psi} - \mathbf{q}^n \rangle \geq 0 \quad \forall \boldsymbol{\psi} \in V^h$$

Here $M_\varepsilon^h(w^n) = \{M_{k,l}^n\}$ is a constant on each edge $e_{k,l}$ approximation of $M_\varepsilon(w^n)$ (defined later).

Although $\nabla \phi$ is not continuous, the product $\mathbf{i}_e \cdot \nabla \phi$ is continuous on every edge e . Hence

$$\langle \nabla \phi, \boldsymbol{\psi} \rangle = \sum_{e \in \mathcal{T}^h} \psi_e \int_e \mathbf{i}_e \cdot \nabla \phi \, de$$

is well defined $\forall \phi \in W_0^h, \boldsymbol{\psi} \in V^h$.

Discrete mixed formulation

Find $w^n \in W_0^h$ and $\mathbf{q}^n \in V^h$ such that

$$\left(\frac{w^n - w^{n-1}}{\tau^n} - f^n, \phi \right) - \langle \nabla \phi, \mathbf{q}^n \rangle = 0 \quad \forall \phi \in W_0^h,$$

$$\langle M_\varepsilon^h(w^n), |\boldsymbol{\psi}| - |\mathbf{q}^n| \rangle + \langle \nabla w^n, \boldsymbol{\psi} - \mathbf{q}^n \rangle \geq 0 \quad \forall \boldsymbol{\psi} \in V^h$$

The discrete mixed formulation can be re-written as

$$s_j \frac{w_j^n - w_j^{n-1}}{\tau^n} + \sum_{e_{j,k} \in \mathcal{E}^h} q_{e_{j,k}}^n - \sum_{e_{k,j} \in \mathcal{E}^h} q_{e_{k,j}}^n = \tilde{f}_j^n,$$

$$|e_{k,l}| M_{k,l}^n (|\psi| - |q_{e_{k,l}}^n|) + (w_l^n - w_k^n) (\psi - q_{e_{k,l}}^n) \geq 0 \quad \forall \psi \in \mathbb{R}^1$$

for every internal node v_j and any edge $e_{k,l}$.

Here $s_j = \int_{\Omega^h} \chi_j$ and $\tilde{f}_j^n = \int_{\Omega^h} f^n \chi_j$, where $\chi_j \in W_0^h$ is the basis function associated with the node v_j : $\chi_j(v_k) = \delta_{j,k}$

The steepest edge descent

In our network approximation, sand (water) from each node flows along the edge of steepest descent.

For any $w \in W_0^h$ we define the steepest edge descent at each node v_k :

$$\partial_{\downarrow}^h w(v_k) := \max \left\{ \frac{w_k - w_l}{|e(k, l)|} : \begin{array}{l} e(k, l) = e_{k, l} \in \mathcal{E}^h \\ \text{or} \\ e(k, l) = e_{l, k} \in \mathcal{E}^h \end{array} \right\}$$

and define M_{ε}^h in two steps.

M_ε^h in nodes and on edges

First, we compute the **node** values, $M_k^n = M_\varepsilon^h(w^n)|_{v_k}$, replacing $|\nabla w_0|$ by $\partial_\downarrow^h w^0$ in M_ε :

$$M_k^n = \begin{cases} k_0 & w_k^n \geq w_k^0 + \varepsilon, \\ k_1(v_k) + (k_0 - k_1(v_k)) \left(\frac{w_k^n - w_k^0}{\varepsilon} \right) & w_k^n \in [w_k^0, w_k^0 + \varepsilon], \\ k_1(v_k) := \max(k_0, \partial_\downarrow^h w^0(v_k)) & w_k^n \leq w_k^0 \end{cases}$$

Second, we define the **edge** values, $M_{k,l}^n = M_\varepsilon^h(w^n)|_{e_{k,l}}$:

$$M_{k,l}^n = \begin{cases} M_k^n & w_k^n \geq w_l^n, \\ M_l^n & \text{otherwise.} \end{cases}$$

The discrete problem is now fully defined.

Two discrete formulations

Our discrete mixed formulation is:

$$s_j \frac{w_j^n - w_j^{n-1}}{\tau^n} + \sum_{e_{j,k} \in \mathcal{E}^h} q_{e_{j,k}}^n - \sum_{e_{k,j} \in \mathcal{E}^h} q_{e_{k,j}}^n = \tilde{f}_j^n,$$

$$|e_{k,l}| M_{k,l}^n (|\psi| - |q_{e_{k,l}}^n|) + (w_l^n - w_k^n) (\psi - q_{e_{k,l}}^n) \geq 0 \quad \forall \psi \in \mathbb{R}^1,$$

for all internal nodes $v_j \in \mathcal{N}_i^h$ and all edges $e_{k,l}$.

The associated with it finite-dimensional analogue of the primal QVI is:

Find $w^n \in K^h(w^n)$ s.t.

$$\sum_{v_j \in \mathcal{N}_i^h} \left(s_j \frac{w_j^n - w_j^{n-1}}{\tau^n} - \tilde{f}_j^n \right) (\phi_j - w_j^n) \geq 0 \quad \forall \phi \in K^h(w^n),$$

where

$$K^h(w) = \left\{ \phi \in W_0^h : \partial_{\downarrow}^h \phi(v_j) \leq M_{\varepsilon}^h(w)|_{v_j} \quad \forall v_j \in \mathcal{N}^h \right\}.$$

Existence and equivalency

Existence of a solution to the discrete primal QVI can be shown using the Brouwer fixed point theorem (or, e.g., as a consequence of Pang and Fukushima 05). For the discrete mixed formulation it follows from

Theorem. If w^n solves the discrete primal QVI then there exists q^n such that $\{w^n, q^n\}$ is a solution to the discrete mixed problem. If a pair $\{w^n, q^n\}$ is a solution to the discrete mixed problem, w^n is a solution to the primal QVI.

Numerical solution

The primal QVI for w^n can be written as an optimization problem with a quadratic functional and implicit constraints that are resolved iteratively, $\frac{w_j^{n,l} - w_k^{n,l}}{|e(j,k)|} \leq M_j^{n,l-1}$, for all nodes v_j and edges $e_{k,j}$ or $e_{j,k}$.

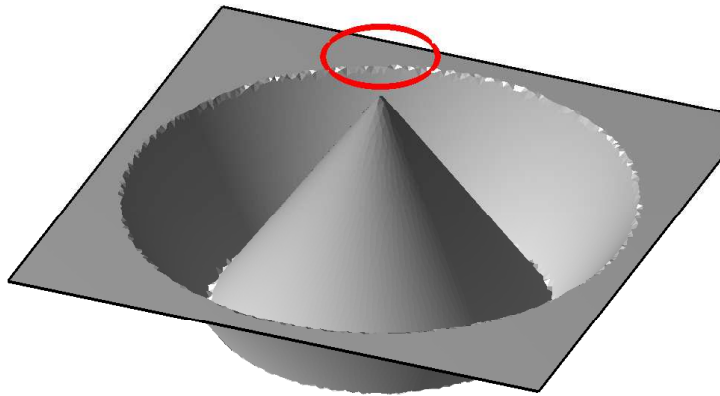
To solve this problem we extended to the QVI case the augmented Lagrangian method with splitting (ALG2 in [Glowinski 84]) by updating the constraints after each iteration). The edge fluxes are determined by the Lagrange multipliers which are computed simultaneously.

Another approach to calculating fluxes: provided w^n was found by a different method, its substitution into the mixed formulation turns that formulation into a linearly constrained L^1 optimization problem. The latter problem can be reformulated (see, e.g., Elad 10) as a linear programming problem and solved efficiently.

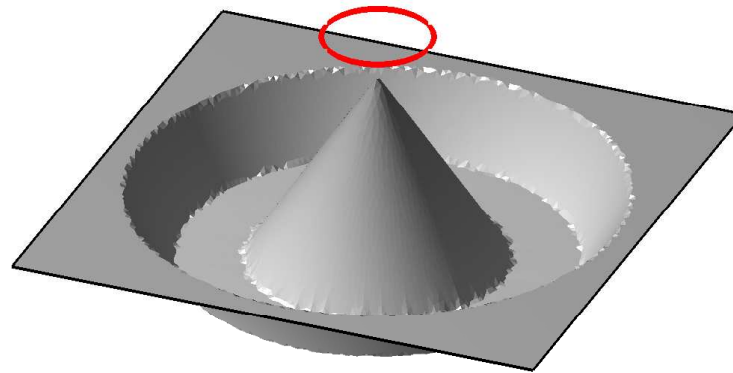
Numerical experiment 1: w^n

A radially symmetric problem with analytical solution.

f is uniform in its support (red line); $\Omega = [-1, 1] \times [-1, 1]$;
regularization parameters: $\varepsilon = 0.01$, $k_0 = 0.005$.



Initial surface



After 50 time steps

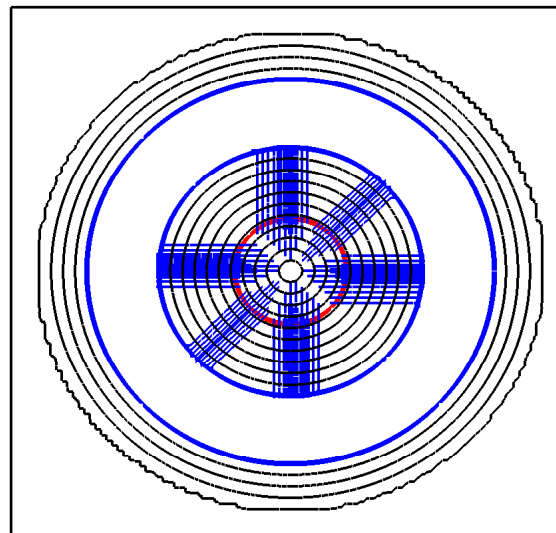
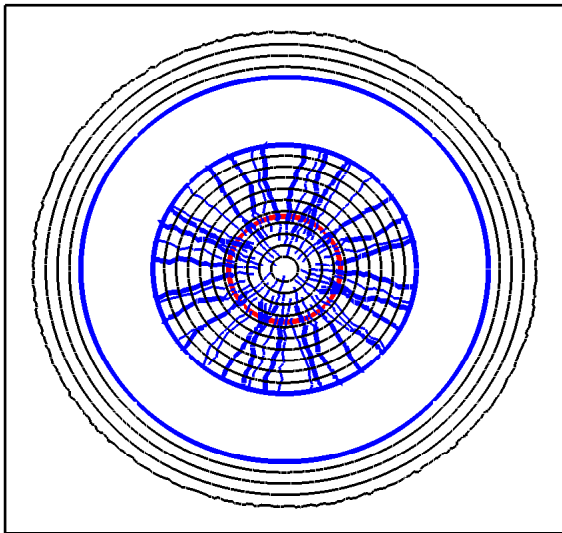
Two meshes: (1) general (irregular) triangulation ($h=0.025$)
and (2) square regular 100×100 grid with SW-NE diagonals;
approximately the same number of triangles.

Error in w^n : less than 0.2% in both cases.

Numerical experiment 1: q^n

Comparing the exact and approximate fluxes is more difficult. Is there at least a weak convergence (as measures) of the singular approximate fluxes to the exact one?

General mesh, $h = 0.025$ Square grid + SW-NE diag.

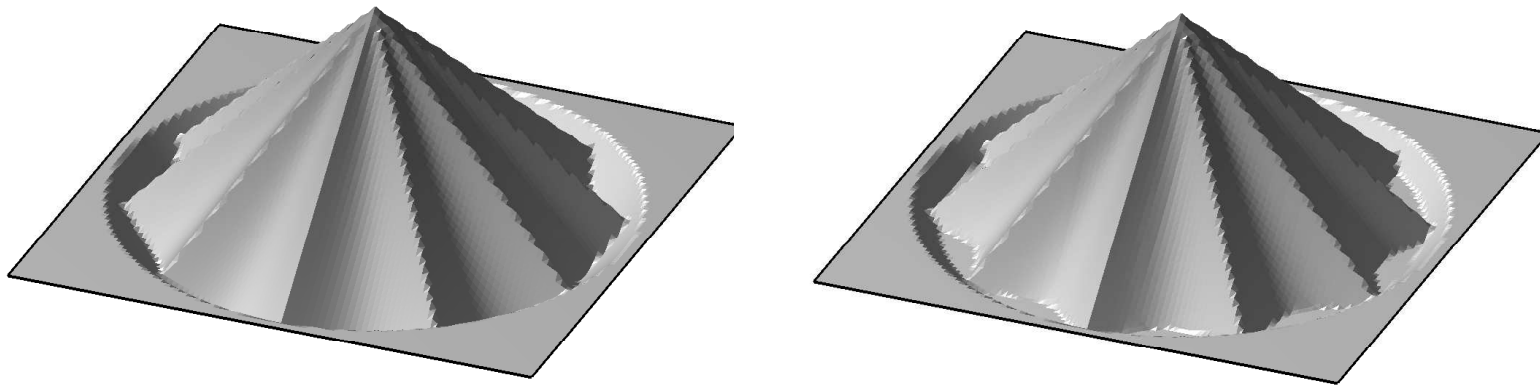


The approximation looks reasonable for the irregular mesh but, in general, the convergence is not guaranteed. This relief lacks such features of natural landscapes as river valleys and, for the regular mesh, the mesh anisotropy prevails.

Numerical experiment 2: w^n

Another artificial landscape: a cone with ten valleys.

Same source, scheme parameters, and the same two meshes;
a smaller time step.



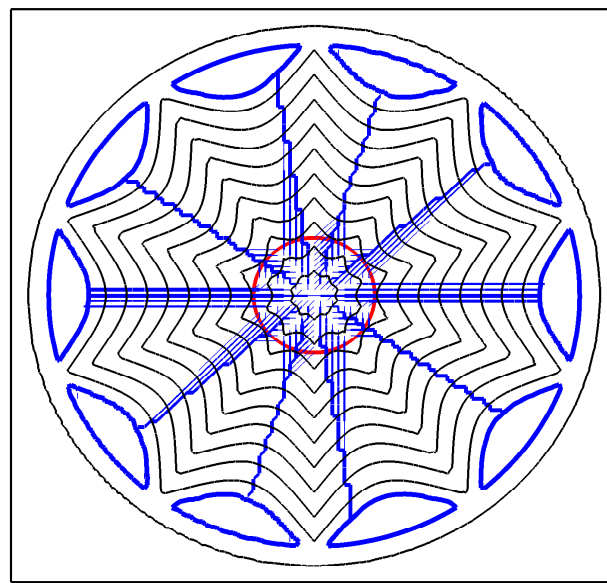
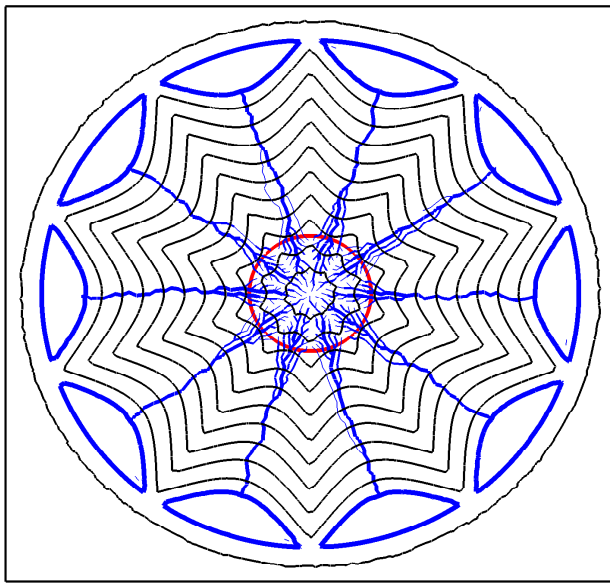
Left: Initial surface;

Right: Computed surface at a time moment before the lakes
merge.

Numerical experiment 2: q^n

The “river valleys” are now pronounced. Although the influence of the regular mesh anisotropy is still obvious, it is much weaker in this case. Most usual lattice methods are affected by the regular grid anisotropy.

General mesh, $h = 0.025$ Square grid + SW-NE diag.



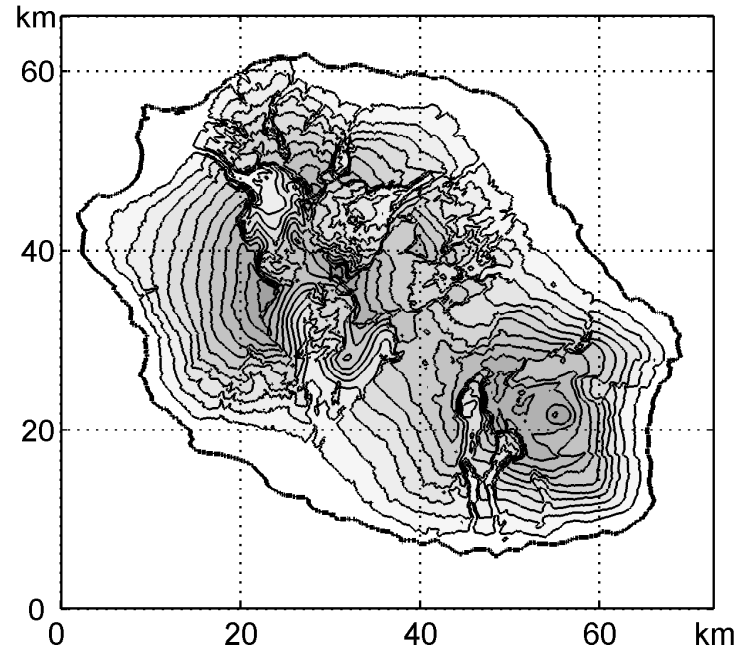
If the river valley is discernible, one can hope to approximate the river by a close to it zig-zag path consisting of edges.

Experiment 3: Réunion island

We used real DEM data for the Réunion island (Fr.), a 63 km long, 45 km wide volcanic island in the Indian ocean. The island has a mountainous relief, its highest point is about 3000m above the sea level.



Satellite image



Topography (from the DEM)

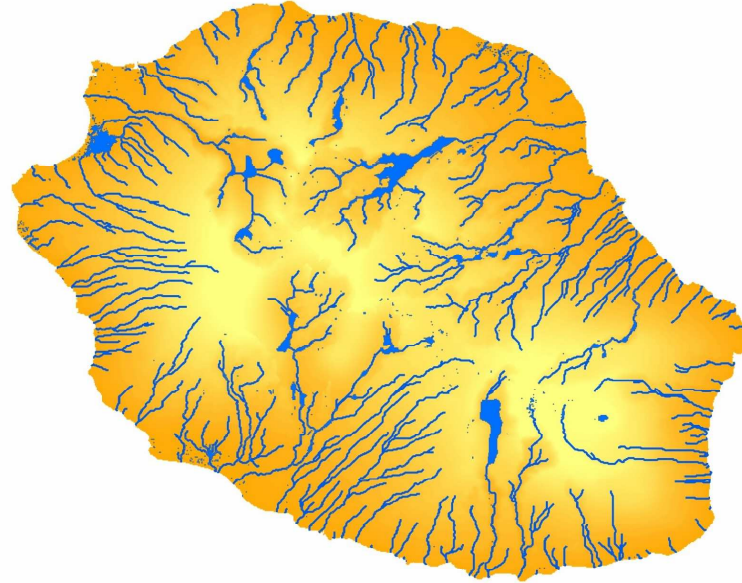
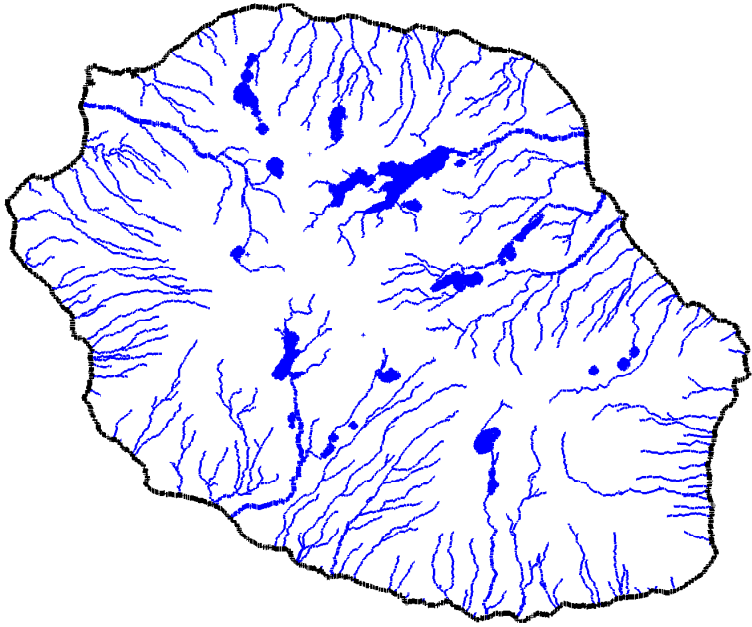
Réunion island DEM

The DEM file contained elevations in the nodes of a regular 736x809 grid (horizontal resolution – 90m, vertical resolution – 1m). The data are from the SRTM (Shuttle Radar Topography Mission) public domain and cover all island.

For numerical simulations we interpolated these elevations into the nodes of an irregular triangulation generated by the Matlab PDE Toolbox. Our finest mesh contained 504 thousand triangles; the horizontal resolution was about 180 m.

The results were compared to the hydrological map produced by Arc Hydro for the same DEM.

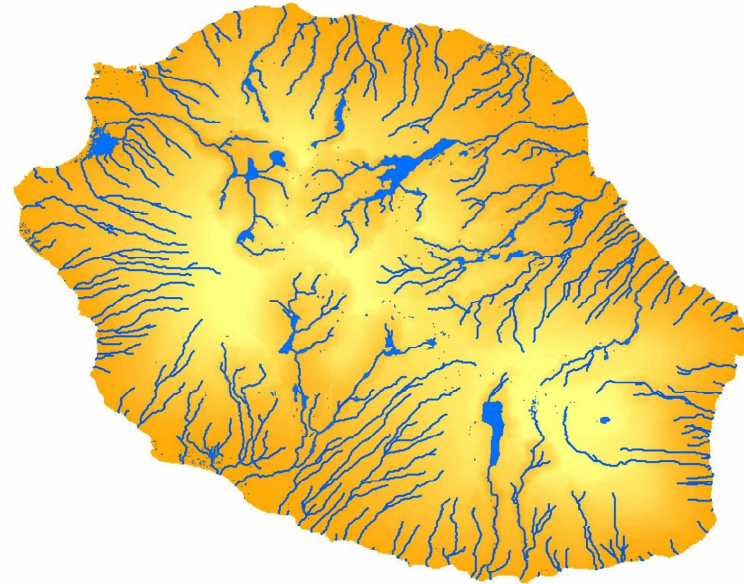
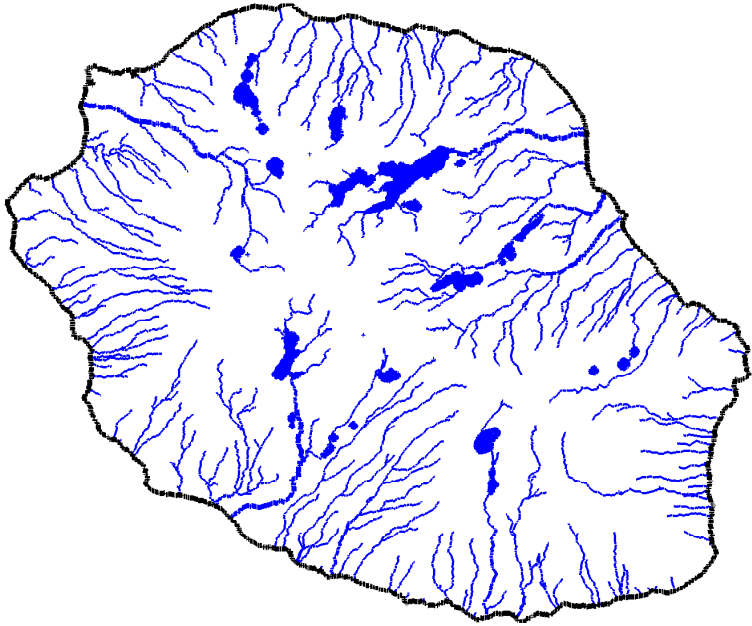
Hydrological maps



Left: our simulation results.

Right: a map produced by the Arc Hydro toolset of ArcGIS.

Hydrological maps



Left: our simulation results.

Right: a map produced by the Arc Hydro toolset of ArcGIS.

Thank you!