Homogenization of mean curvature motions

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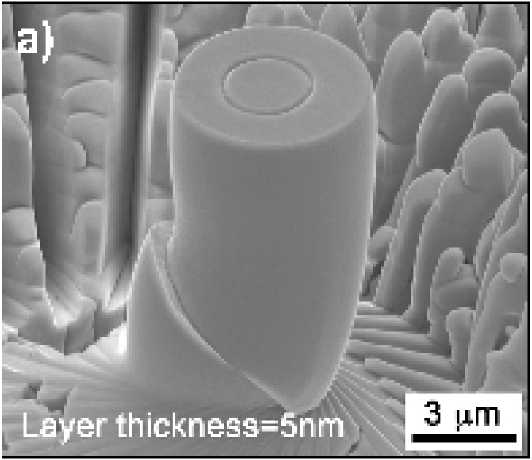
Frauenchiemsee; June 14, 2012

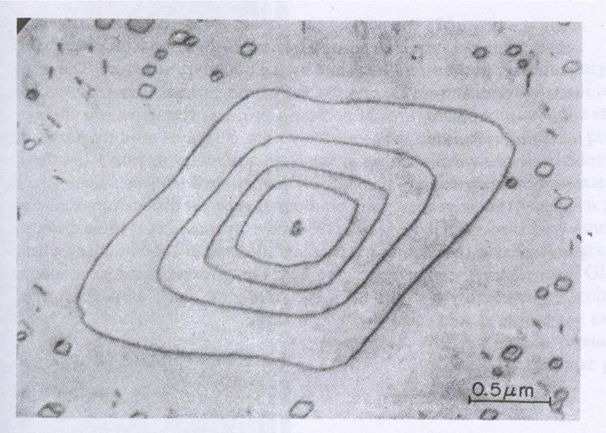
joint work with L.A. Caffarelli

Motivation

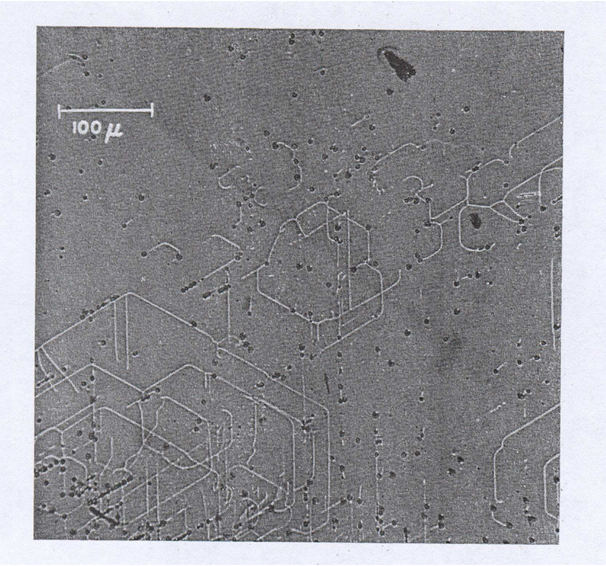
Visco-plasticity



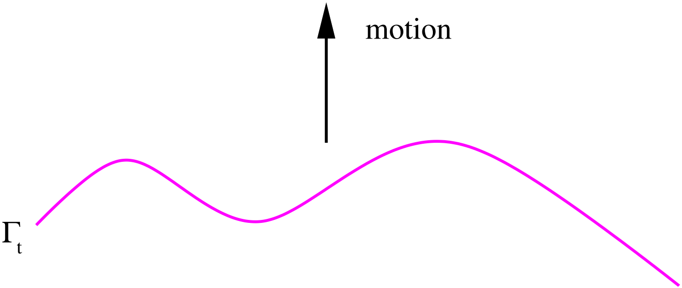
Dislocation lines in crystals



Precipitates = obstacles

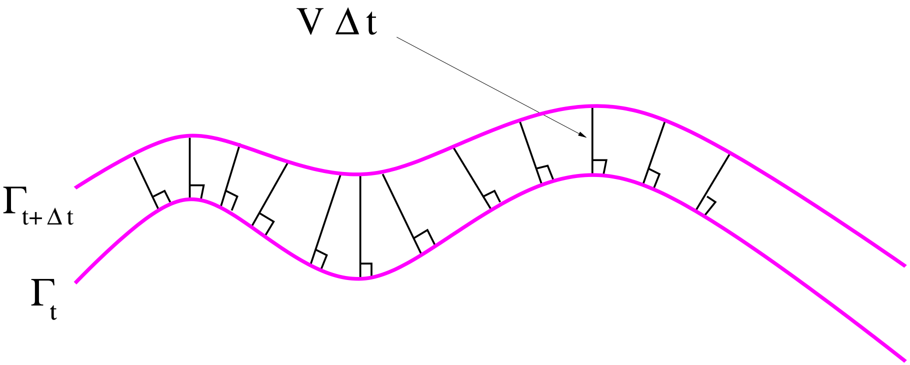


Forced mean curvature motion in $\mathbb{R}^N$



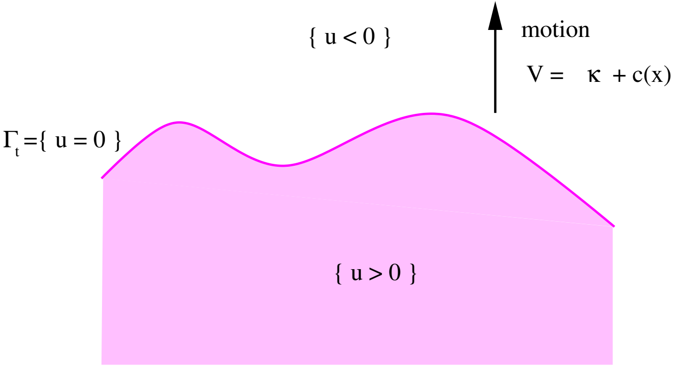
$$\begin{cases} V & = & \text{normal velocity} & = \kappa + c(x) \\ \kappa & = & \text{mean curvature} & = \displaystyle\sum_{i=1}^{N-1} \kappa_i \\ c(x) & = & \mathbb{Z}^N\text{-periodic forcing} & \end{cases}$$

Motion with normal velocity



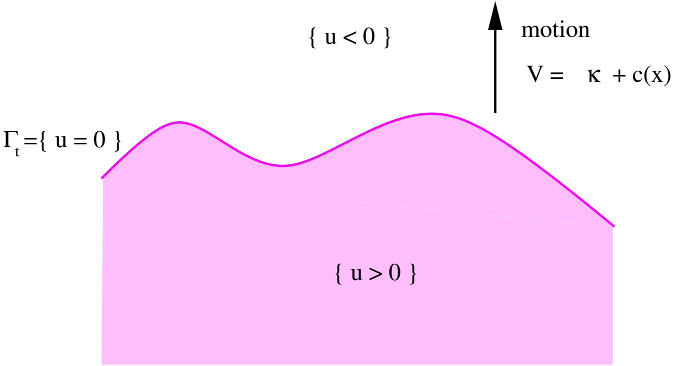
Change of topology

Level sets formulation



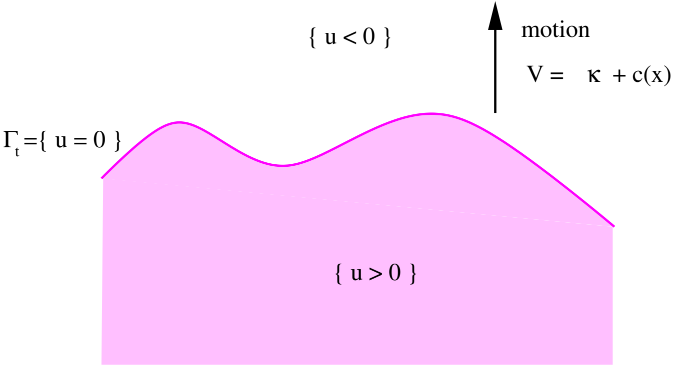
$$u_t = F(D^2u, Du, x)$$

Level sets formulation



$$u_t = F(D^2u, Du, x) := \text{trace} \left\{ D^2u \cdot \left(I - \frac{Du}{|Du|} \otimes \frac{Du}{|Du|} \right) \right\} + c(x)|Du|$$

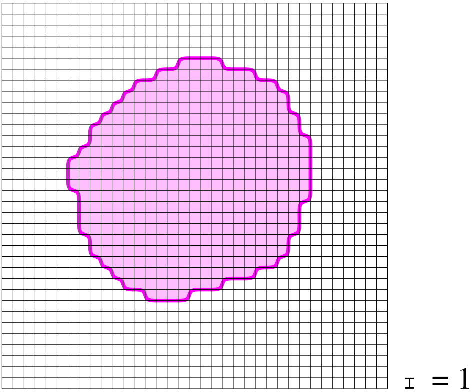
Level sets formulation



$$u_t = F(D^2u, Du, x) := \text{trace} \left\{ D^2u \cdot \left(I - \frac{Du}{|Du|} \otimes \frac{Du}{|Du|} \right) \right\} + c(x)|Du|$$

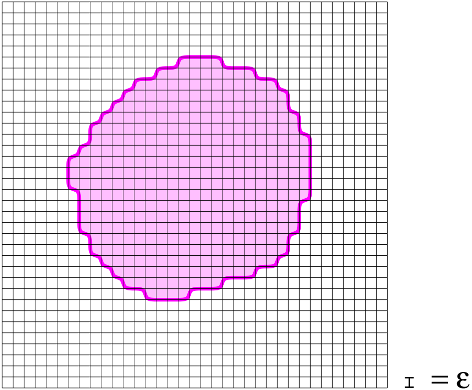
[Chen, Giga, Goto, '91]
[Evans, Spruck, '91]

Homogenization at large scale ?

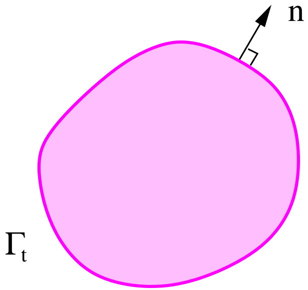


$$V = \kappa + c(x)$$

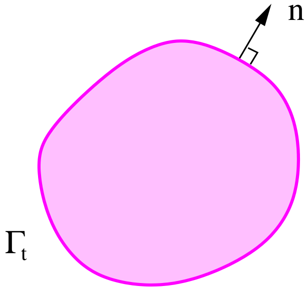
Rescaling



$$V^\varepsilon = \varepsilon\kappa + c\left(\frac{x}{\varepsilon}\right)$$



$$V^\varepsilon \quad \longrightarrow \quad V^0 = \bar{c}(n) \qquad \text{with } n \;=\; \text{normal to } \Gamma_t$$



$$V^{\varepsilon} \quad \longrightarrow \quad V^0 = \bar{c}(n) \qquad \text{with } n \ = \ \text{normal to } \Gamma_t$$

Find $\bar{c}(n)$ and v bounded such that

$$\bar{c}(n) = F(D^2 v, n + Dv, x)$$

Related literature

- motivations
  [Barles, Soner, Souganidis, '93] limits of reaction-diffusion equations
  [Craciun, Bhattacharya, '04] dislocations in crystals
- case $N = 2$ and $c = c(x_1) > 0$
  [Chen, Namah, '97]
  [Lou, Chen, '09]
- cases with c non positive
  [Dirr, Karali, Yip, '08] c small
  [Cardaliaguet, Lions, Souganidis, '09] $N = 2$, $c = c(x_1)$
  [Cesaroni, Novaga, '11]
  $N \geq 2$, $c = c(x')$ with $x' = (x_1, ..., x_{N-1})$, pseudo correctors
- pinning $\bar{c} = 0$
  [Caffarelli, De La Llave, '01]
  [Chambolle, Thouroude, '09]

Related literature

- geometric motion
  [Lou, '07] $N = 2$, $V = c(\kappa, n, x_1)$
  [Lions, Souganidis, '05]
  $N \geq 2$, $V = \kappa + c(x)b(n)$, $b \geq 1$, $c^2 > (N-1)|Dc|$
- other scalings
  [Barles, Cesaroni, Novaga, '11] $V = \kappa + \varepsilon^{-1}c(\varepsilon^{-1}x')$ with $\int c = 0$
  [Cesaroni, Novaga, Valdinoci, '11] $V = \kappa + c(\varepsilon^{-1}x')$, $N = 2$

$$\bar{c} = a : D^2 u + c(x)|Du| \quad \text{with} \quad u(x) = n \cdot x + v(x)$$

We look at the $\max \ |w|$ with $w = Du$:

Condition $c^2 > (N-1)|Dc|$ for $V = \kappa + c(x)$

$$\bar{c} = a : D^2u + c(x)|Du| \quad \text{with} \quad u(x) = n \cdot x + v(x)$$

We look at the $\max\ |w|$ with $w = Du$:

$$\begin{aligned} 0 &= w\left(a : D^2 w \right. && \left. + Dc\ |w|\right) \quad +0 \\ &\le a : \left(D^2\left(\frac{w^2}{2}\right) - (Dw)(Dw)^T\right) && +|Dc|w^2 \\ &\le -a : (D^2u)^2 && +|Dc|w^2 \end{aligned}$$

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Using

$$\left(\sum_1^{N-1} \lambda_i \right)^2 \leq (N-1) \left(\sum_1^{N-1} \lambda_i^2 \right)$$

we get

$$\frac{1}{N-1} (\bar{c} - c|w|)^2 \leq |Dc| w^2$$

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Using

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we get

$$\frac{1}{N-1} (\bar{c} - c|w|)^2 \leq |Dc| w^2$$

$$|w| \to +\infty \qquad \Longrightarrow \qquad c^2 \leq (N-1)|Dc|$$

Main result

$$V = \kappa + c(x)$$

- Homogenization if $c^2 > (N-1)|Dc|$
  [Lions, Souganidis, '05]

- Homogenization if $c > 0$?
  [Caffarelli, M., '12]
    -
    -

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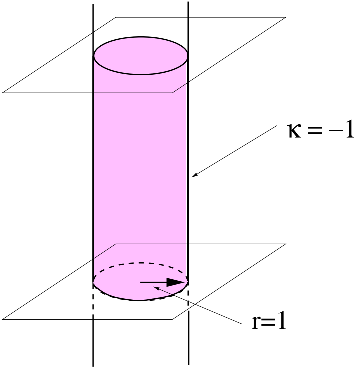
- Homogenization if $c > 0$?
  [Caffarelli, M., '12]
  - No if $N \geq 3$
  - Yes if $N = 2$

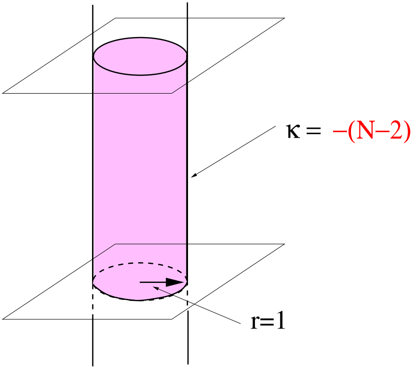
Counter-example in dimension $N \geq 3$ with $c > 0$

Stationary cylinder in dimension $N = 3$



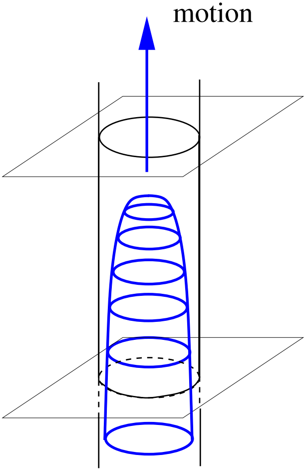
stationary solution of $V = \kappa + 1$

Stationary cylinder in dimension $N \geq 3$



stationary solution of $V = \kappa + (N-2)$

Interior cylindrical solution

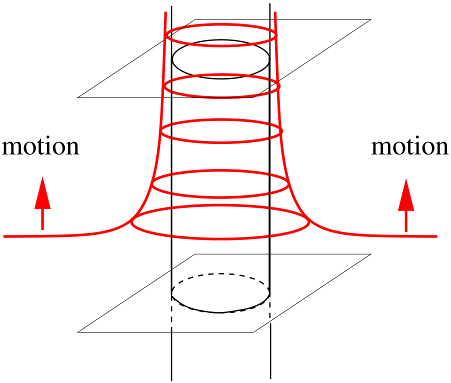


"finger" solution of $V = \kappa + c(|x'|)$ with $c(r=1) = N-2$

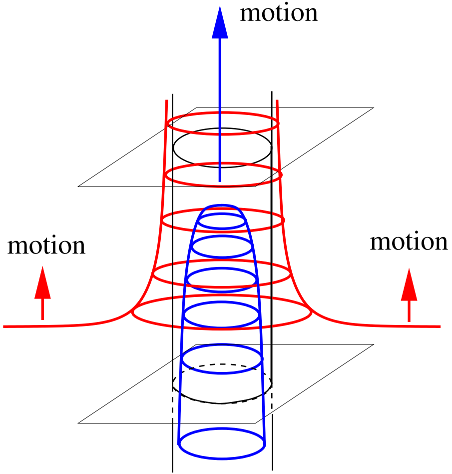
with

$$x' = (x_1, ..., x_{N-1})$$

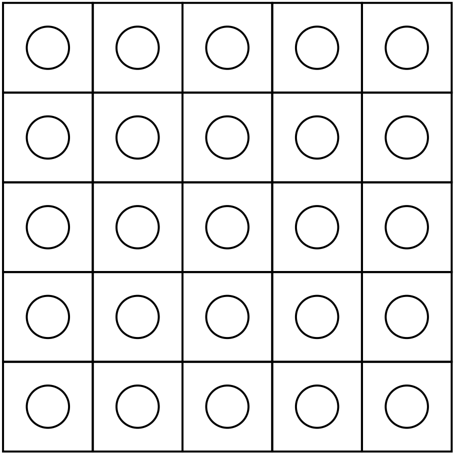
Exterior cylindrical solution



Exterior/interior solutions

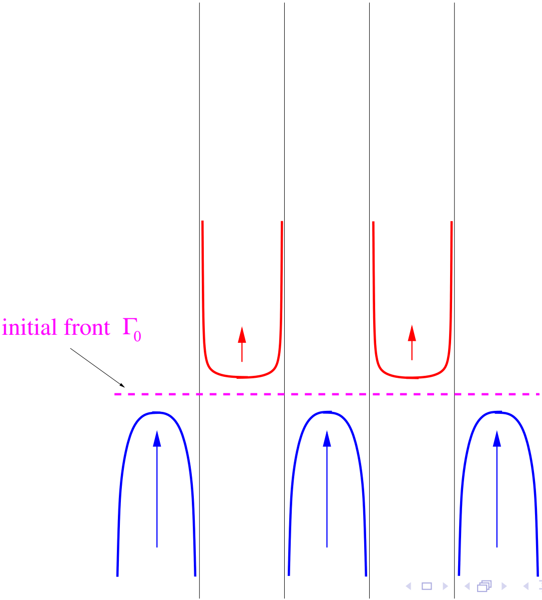


Repeat periodically the patern

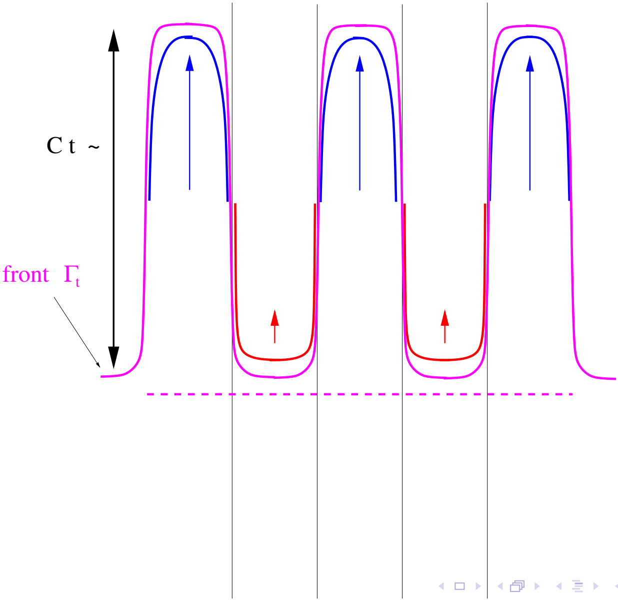


periodic $c(x')$

Initial cross section

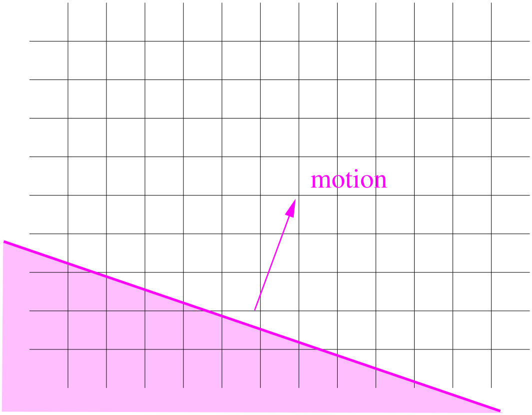


Cross section after a long time t

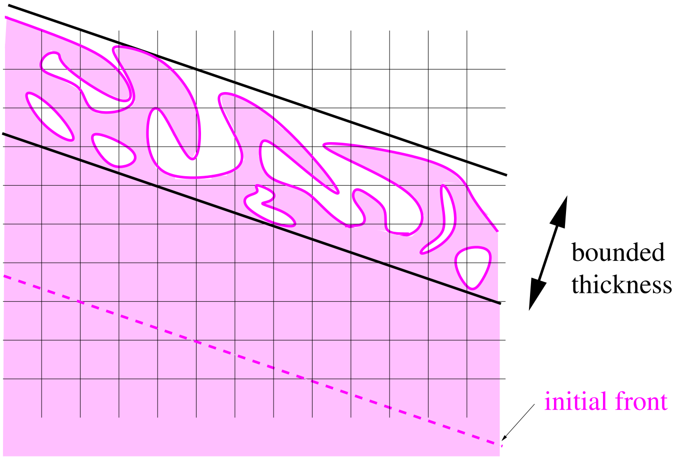


Homogenization in dimension $N = 2$ with $c > 0$

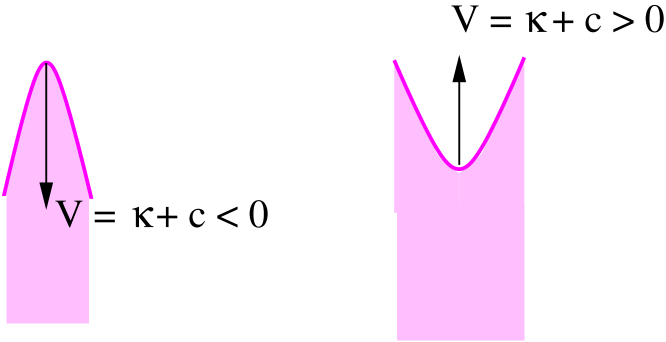
The goal



The goal



Fact 1 : monotonicity of the front for $c > 0$



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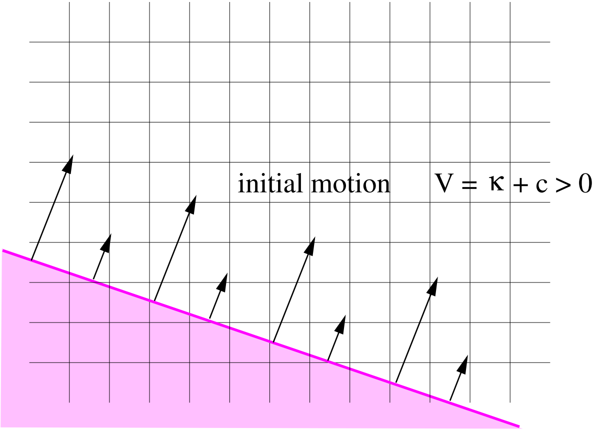
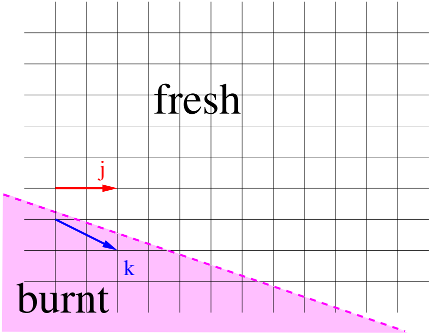


$$\Longrightarrow \qquad V > 0 \quad \text{for all time } t \geq 0$$

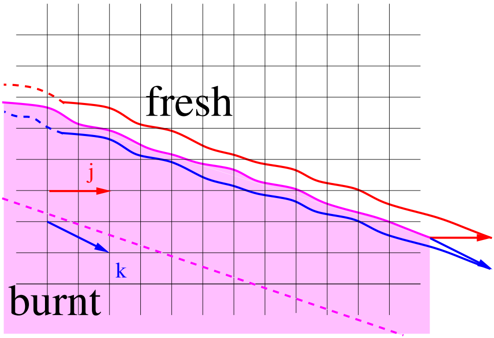
- front of a fire : the front never comes back
- connectedness of the burnt region

Fact 2 : the Birkhoff property = integer translations



$$\begin{cases} \text{burnt} + k & \subset \text{burnt} \\ \text{fresh} + j & \subset \text{fresh} \end{cases}$$

Fact 2 : the Birkhoff property = integer translations



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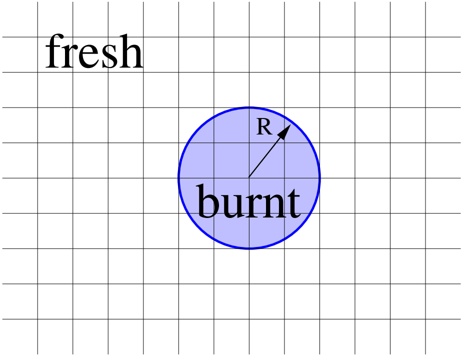
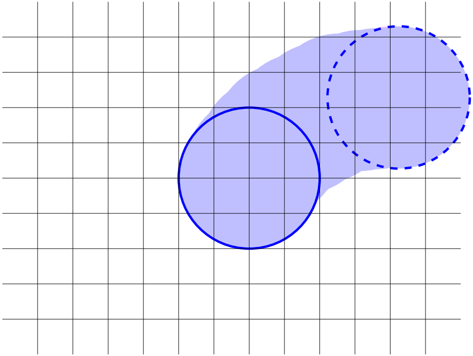
Fact 3 : the burnt ball subsolution



R large s.t. $V = \kappa + c \geq \delta > 0$

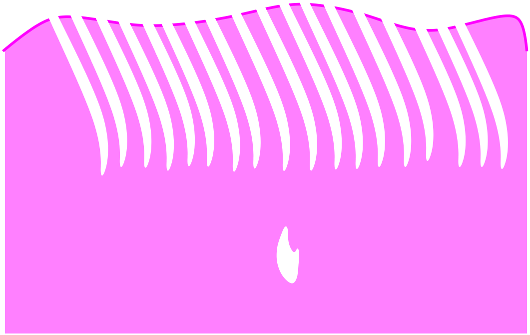
Fact 3 : the burnt ball subsolution



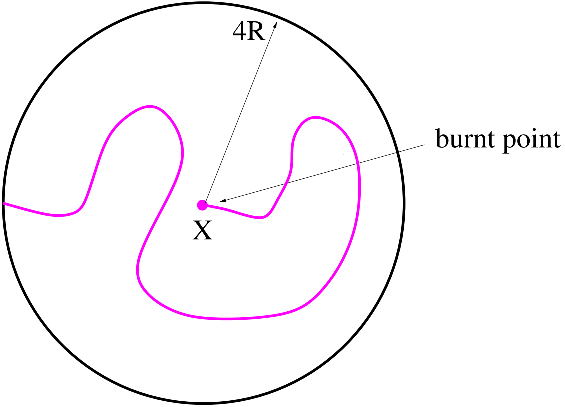
R large s.t. $V = \kappa + c \geq \delta > 0$

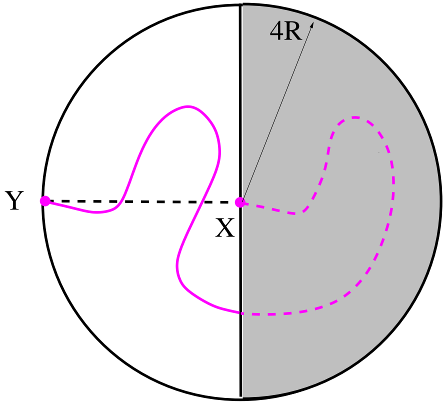
move the ball at a velocity $\leq \delta$

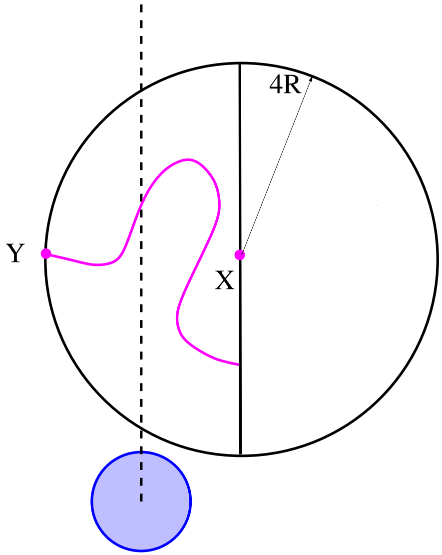
Typical situation to exclude

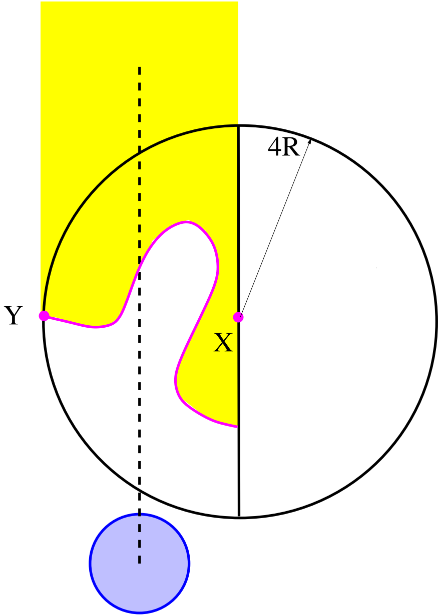


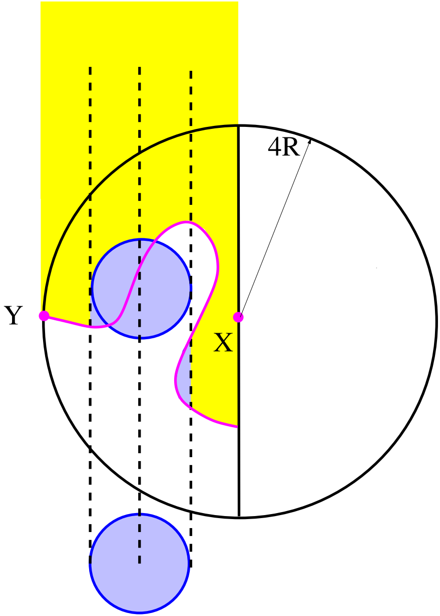
Proof of the bounded thickness of the front

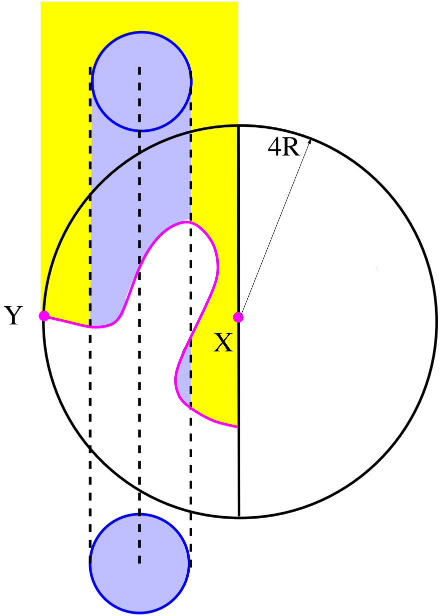

4R
burnt point
X

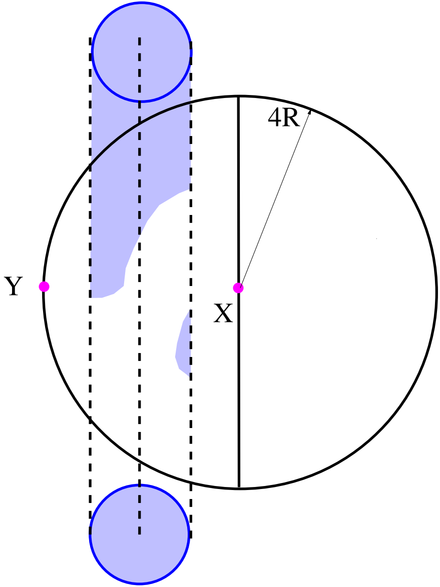

4R
Y
X

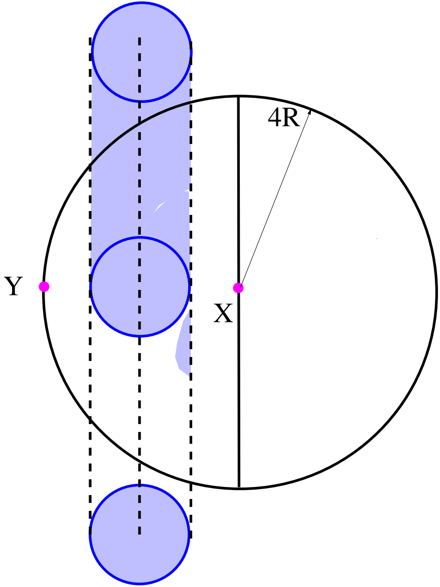

4R
Y
X

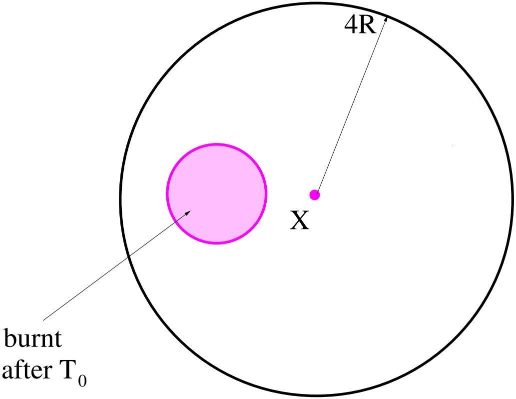

4R
Y
X

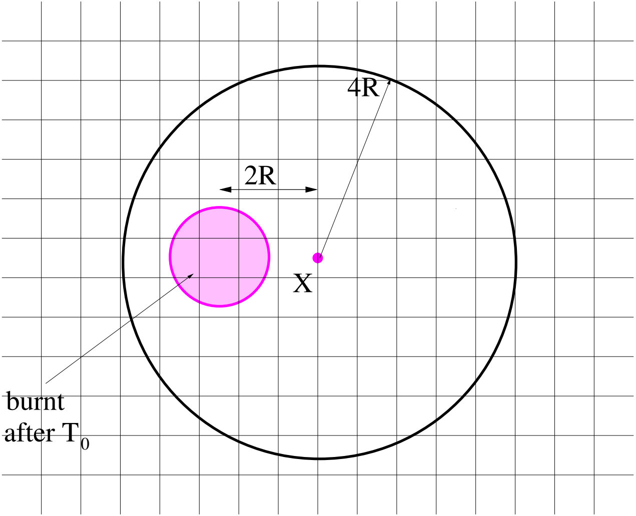

4R
Y
X

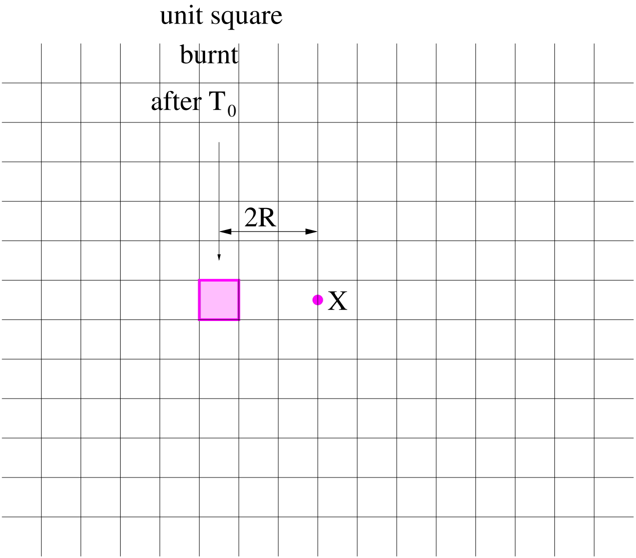

4R
Y
X

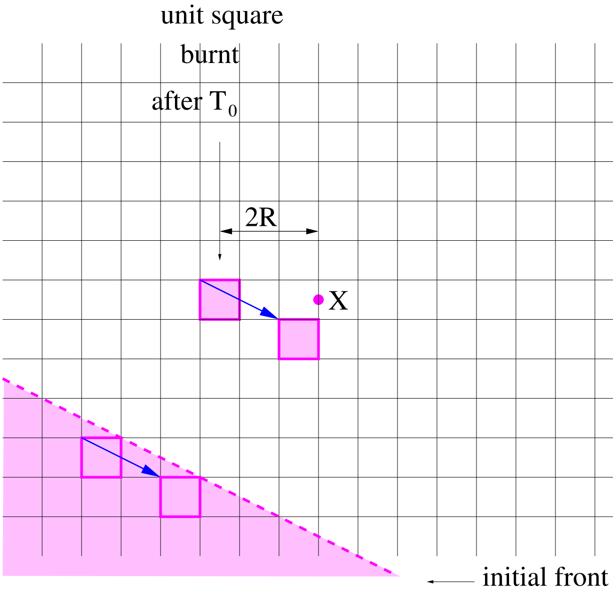

4R
Y
X

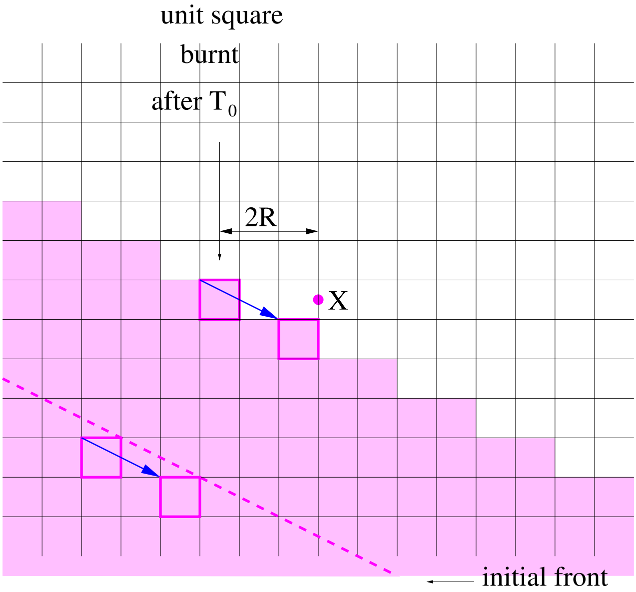

4R
Y
X

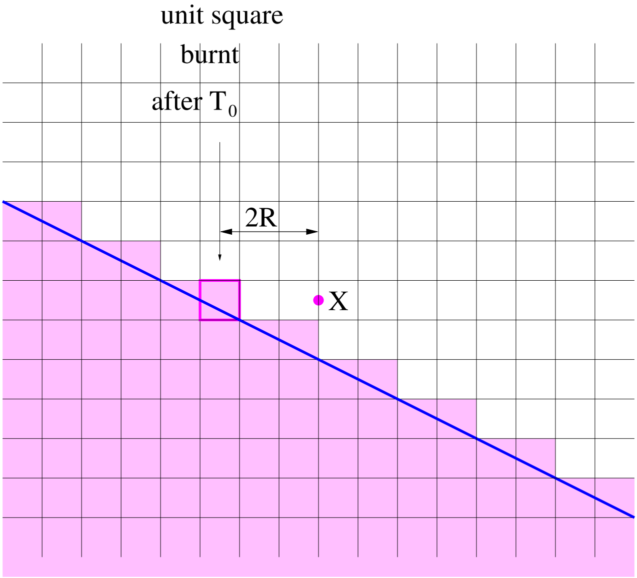

4R
X
burnt
after T_0

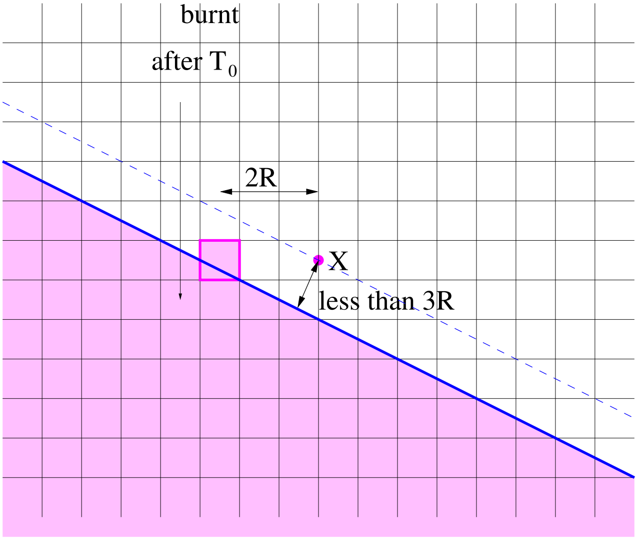

4R
2R
X
burnt
after T_0


unit square
burnt
after T_0
2R
X

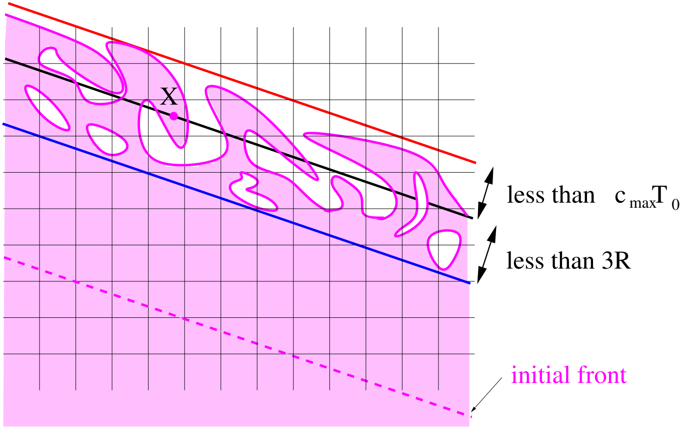

unit square
burnt
after T_0
2R
X
initial front


unit square
burnt
after T_0
2R
X
initial front


unit square
burnt
after T_0
2R
X


burnt
after T_0
2R
X
less than 3R

End of the proof



Warning for experts

Evans perturbed test function method
does not apply to geometric motions

Generalizations in dimension $N = 2$

Assumptions (A)

- (A1) periodic dynamics

$$V = c(\kappa, n, x) \quad \text{with} \quad \begin{cases} c: & \mathbb{Z}^2\text{-periodic in } x, \\ + & \text{standard assumptions on } c \end{cases}$$

- (A2) straight line subsolutions

$$c_{max} \geq \quad c(0, n, x) \quad \geq c_{min} > 0$$

Let

$$V^\varepsilon = c\left(\varepsilon\kappa, n, \frac{x}{\varepsilon}\right)$$

Thm (Homogenization in 2D) [Caffarelli, M.]

Under assumption (A), we have

$$V^\varepsilon \to V^0 = \bar{c}(n) \geq \quad c_{min} > 0 \qquad \text{as } \varepsilon \to 0$$

Dynamics in 2D

Let

$$V^\varepsilon = c\left(\varepsilon\kappa, n, \frac{x}{\varepsilon}\right)$$

Thm (Homogenization in 2D) [Caffarelli, M.]

Under assumption (A), we have

$$V^\varepsilon \to V^0 = \bar{c}(n) \geq \quad c_{min} > 0 \qquad \text{as } \varepsilon \to 0$$

$$\bar{c}(n) = F(D^2 v, n + Dv, x)$$

with

$$\sup v - \inf v \quad \leq \quad 100 \; R \frac{c_{max}}{c_{min}}$$

Assumptions (A')

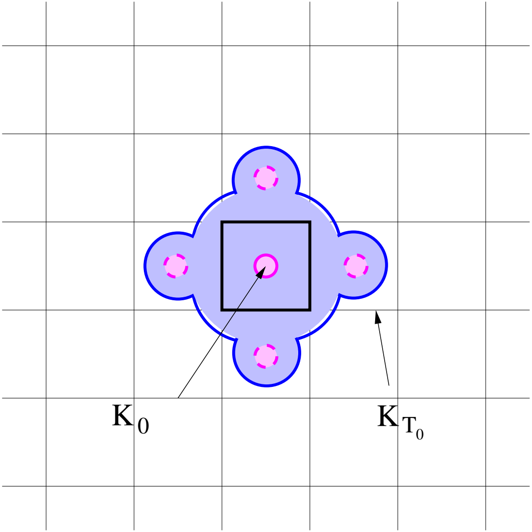
- (A1) periodic dynamics

$$V = c(\kappa, n, x) \quad \text{with} \quad \begin{cases} c: & \mathbb{Z}^2\text{-periodic in } x, \\ + & \text{standard assumptions on } c \end{cases}$$

- (A2') expanding subsolution
  There exists a family of compact sets $(K_t)_{t\geq 0}$ which is a subsolution and

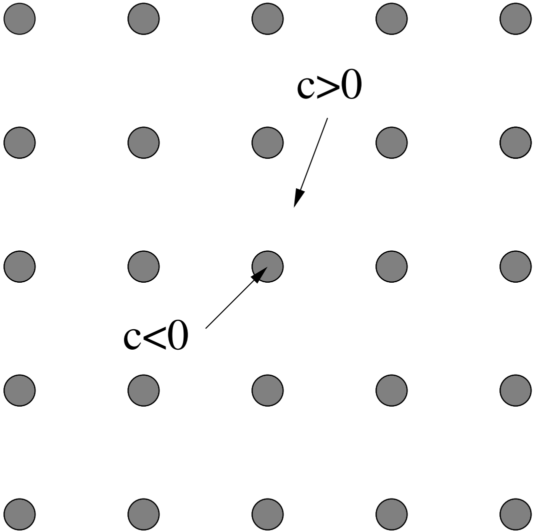
$$\begin{cases} K_t = K_{T_0} \supset [0,1]^2 \quad \text{for all} \quad t \geq T_0 > 0, \\ \\ K_{T_0} \supset K_0 + e \quad \text{for all} \quad e = 0, \pm e_1, \pm e_2 \end{cases}$$

Illustration of (A2') : an expanding subsolution

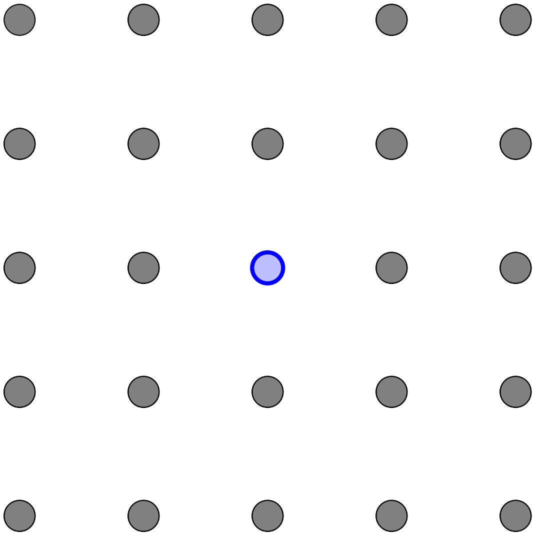


An example

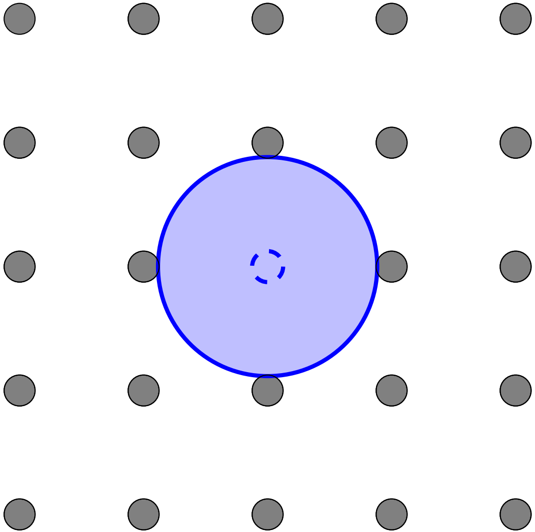
Example for $V = \kappa + c(x)$ with sign changing c



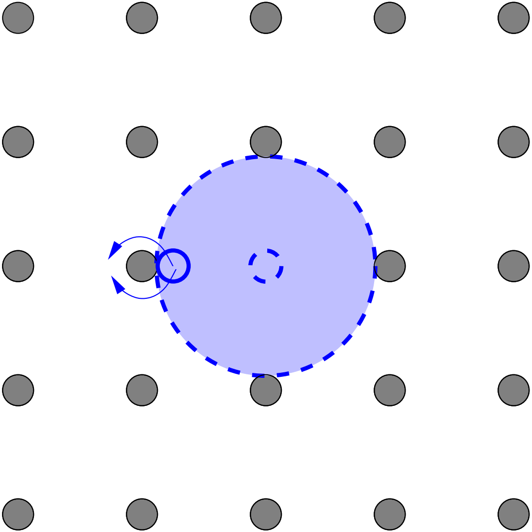
Example for $V = \kappa + c(x)$ with sign changing c



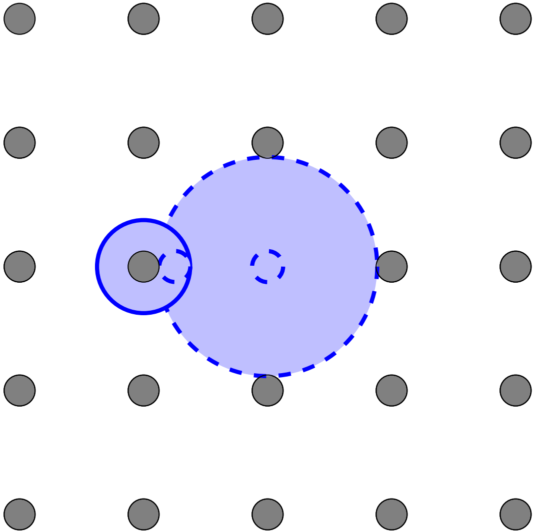
Example for $V = \kappa + c(x)$ with sign changing c



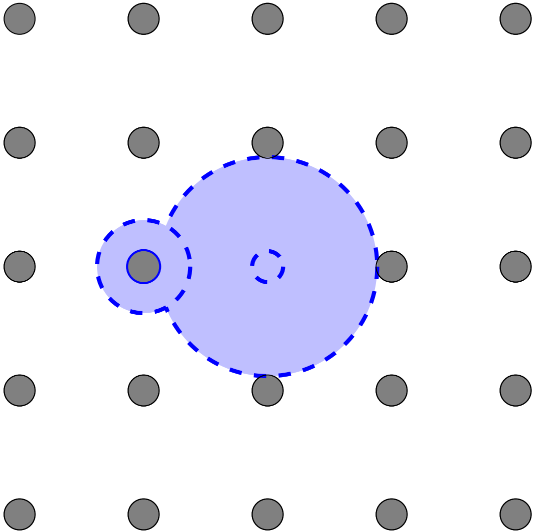
Example for $V = \kappa + c(x)$ with sign changing c



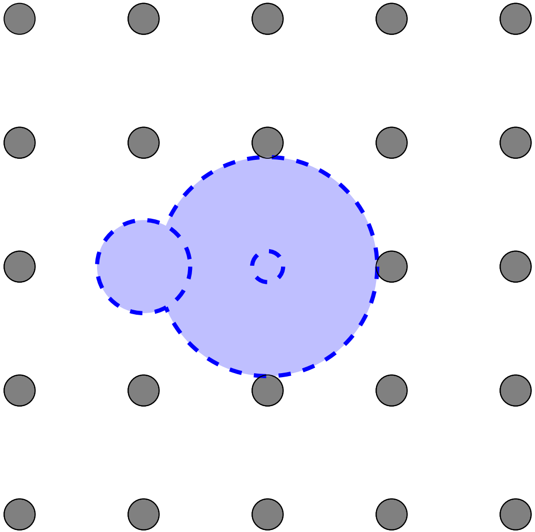
Example for $V = \kappa + c(x)$ with sign changing c



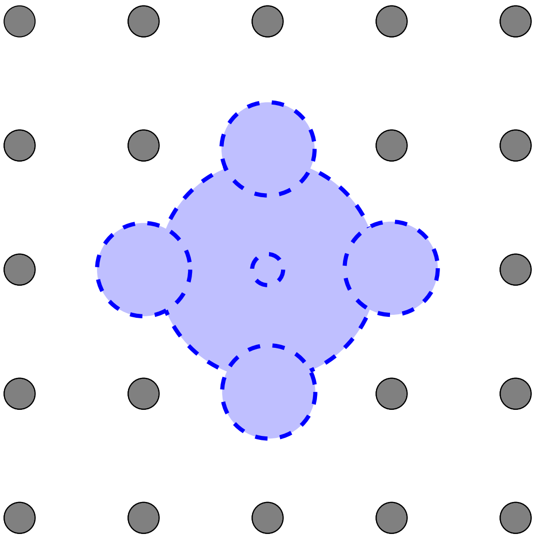
Example for $V = \kappa + c(x)$ with sign changing c



Example for $V = \kappa + c(x)$ with sign changing c



Example for $V = \kappa + c(x)$ with sign changing c



Thank you for your attention