# Global weak solutions for an incompressible, Newtonian fluid interacting with a linearly elastic shell

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# Outline

# Koiter's energy

**Koiter's energy for thin, linearly elastic shells (transverse displacements)**

$$K(\eta) = \frac{1}{2} \int_{\partial\Omega \setminus \Gamma} \epsilon \langle C, \sigma(\eta) \otimes \sigma(\eta) \rangle + \frac{\epsilon^3}{3} \langle C, \theta(\eta) \otimes \theta(\eta) \rangle \; dA$$

Shell elasticity tensor $C_{\alpha\beta\gamma\delta} = \frac{4\lambda\mu}{\lambda+2\mu} g_{\alpha\beta} g_{\gamma\delta} + 2\mu(g_{\alpha\gamma} g_{\beta\delta} + g_{\alpha\delta} g_{\beta\gamma})$

Linearisations $\sigma$ and $\theta$ at $\eta = 0$ of the

- change of metric tensor $\Sigma_{\alpha\beta}(\eta) = \frac{1}{2}(g_{\alpha\beta}(\eta) - g_{\alpha\beta})$
- change of curvature tensor $\Theta_{\alpha\beta}(\eta) = h_{\alpha\beta}(\eta) - h_{\alpha\beta}$

# Koiter's energy

## Koiter's energy for thin, linearly elastic shells (transverse displacements)

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Taking into account inertia, the equations of motion take the form

## Generalisation of the Kirchhoff-Love plate equation

$$\epsilon \rho_S \partial_t^2 \eta + \operatorname{grad}_{L^2} K(\eta) = g \text{ in } I \times \partial\Omega \setminus \Gamma.$$

# Coupled system

## Equations of the coupled fluid-shell system

$$\rho_F(\partial_t \mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u}) = \operatorname{div}(2\sigma D\mathbf{u} - \pi \mathrm{id}) + \mathbf{f} \qquad \text{in } \Omega_{\eta(t)}$$

$$\operatorname{div} \mathbf{u} = 0 \qquad \text{in } \Omega_{\eta(t)}$$

$$\epsilon \rho_S \partial_t^2 \eta + \operatorname{grad}_{L^2} K(\eta) = ((-2\sigma D\mathbf{u} + \pi \mathrm{id})\boldsymbol{\nu}_t) \cdot \boldsymbol{\nu} + g \qquad \text{in } \partial\Omega \setminus \Gamma$$

$$\mathbf{u} = \partial_t \eta \boldsymbol{\nu} \qquad \text{on } \partial\Omega_{\eta(t)} \setminus \Gamma$$

$$\mathbf{u} = 0 \qquad \text{on } \Gamma$$

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Some important results:

**Beirão da Veiga (2004):** Local existence of strong solutions in plate case with added damping of the shell equation in 2d for small data

**Chambolle, Desjardins, Esteban, Grandmont (2005):** Global existence of weak solutions in plate case with added damping

**Grandmont (2008):** Same without damping

# Energy estimates

## Energy identity

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_{\eta(t)}} |\mathbf{u}|^2 \, dx + \frac{1}{2}\frac{d}{dt}\int_{\partial\Omega\setminus\Gamma} |\partial_t \eta|^2 \, dA + \frac{d}{dt}K(\eta)$$
$$= -\int_{\Omega_{\eta(t)}} |\nabla \mathbf{u}|^2 \, dx + \int_{\Omega_{\eta(t)}} \mathbf{f}\cdot\mathbf{u} \, dx + \int_{\partial\Omega\setminus\Gamma} g \, \partial_t \eta \, dA$$

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By Gronwall's lemma:

$$\eta \in W^{1,\infty}(I, L^2(\partial\Omega\setminus\Gamma)) \cap L^\infty(I, H_0^2(\partial\Omega\setminus\Gamma))$$
$$\mathbf{u} \in L^\infty(I, L^2(\Omega_{\eta(t)})) \cap L^2(I, H^1(\Omega_{\eta(t)}))$$

Note: $W^{1,\infty}(L^2) \cap L^\infty(H^2) \hookrightarrow C^{0,\mu}(C^{0,1-2\mu})$, $0 < \mu < 1/2$

# Strategy of proof

**Basic strategy**:

- Decouple the equations from the moving domain
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**Hard to prove compactness**:

- Low regularity of the domain
- Hard to use difference quotients, impossible to use Aubin-Lions
- No Bochner spaces
- No substitute for Bochner spaces of dual space-valued functions
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Idea: Give an elementary proof of Aubin-Lions theorem

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1. Choose $\varphi \in H^1_0(\Omega)$ as a test function:

$$\int_0^t \int_\Omega \nabla u_n \cdot \nabla \varphi \, dxdt = \int_\Omega u_n(t) \, \varphi \, dx - \int_\Omega u_n(0) \, \varphi \, dx =: c_{\varphi,n}(t) - c_{\varphi,n}(0)$$

$\Rightarrow (c_{\varphi,n})$ is bounded in $C^{0,\frac{1}{2}}(\bar{I})$, uniformly in $n$ and $\|\varphi\|_{H^1_0(\Omega)} \leq 1$

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4. Ehrling lemma argument:

$$\sup_{\|\varphi\|_{L^2(\Omega)} \leq 1} \int_I (c_{\varphi,n} - c_\varphi) \, dt \leq \epsilon \, \|u_n - u\|_{L^2(I, H_0^1(\Omega))} + c(\epsilon) \sup_{\|\varphi\|_{H_0^1(\Omega)} \leq 1} \int_I (c_{\varphi,n} - c_\varphi) \, dt.$$

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We conclude: $u_n \longrightarrow u$ in $L^2(I, L^2(\Omega))$.

# Thank you for your attention!