Instantaneous Control of the Cahn-Hilliard Navier-Stokes system

Christian Kahle
joint work with
Michael Hinze

12.06.2012

Aims of Control

- Steering a time dependent system into a 'desired' state.
- Stabilizing of instable states.
- Stabilizing against disturbance.

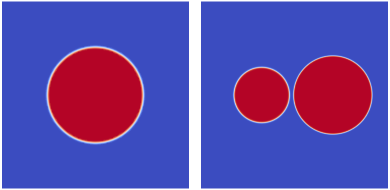


Figure: Morphing a bubble into two.

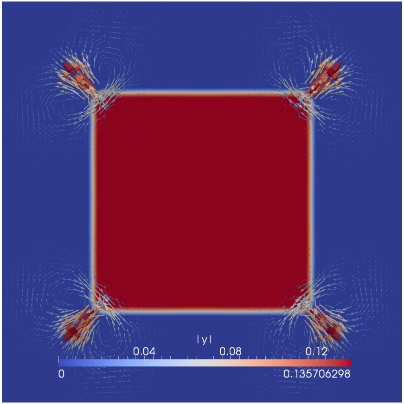


Figure: Stabilizing an instable state.

Outline

Outline

General control concept

$x(t)$ state of some system governed by pdes, A linear operator, $b(x,t)$ nonlinearity, x_d given desired state

$$\dot{x}(t) + Ax = b(x,t) + u$$

AIM: steer $x \to x_d$, $t \to \infty$ using control $u = K(x,t)$, evaluation of u should be fast.

Instantaneous control

Time discretization $t^{k+1} = t^k + \tau$ and calculate $\left(u^{k+1}, \ldots, u^{k+L}\right)$ by minimizing

$$J_k = \sum_{i=1,\ldots,L} \frac{1}{2}\|x^{k+i} - x_d^{k+i}\|^2 + \frac{\alpha}{2}\|u^{k+i}\|^2$$

and set $u = u^{k+1}$.
Approximation of u^{k+1} by **one gradient step** with suitable initial control u_0^{k+1} and step size $\rho > 0$.
Here: $L = 1$, $\rightarrow$ drawback: No sufficient consideration of future behaviour!

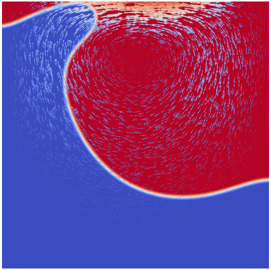
Outline

Setting

Let $\Omega \subset \mathbb{R}^2$ be a spatial domain filled with a fluid containing two phases A and B with respective concentrations c_A and c_B.
Let

$$c = \frac{c_A - c_B}{c_A + c_B}.$$

denote the order parameter satisfying $-1 \leq c \leq 1$ and $c = 1$ in the pure A phase and $c = -1$ in the pure B phase.



The flow y inside Ω might be driven by boundary values or volume forces and is influenced by the dynamics at the interface between A and B.

The Cahn-Hilliard Navier-Stokes system

We consider for mean flow $y = \frac{1+c}{2} y_A + \frac{1-c}{2} y_B$

y : flowfield, p : pressure, c : concentration, w : chemical potential

$$
\begin{aligned}
&(y_t, v) + \frac{1}{Re}(\nabla y, \nabla v) + (y \nabla y, v) \\
&\qquad +(p, \operatorname{div} v) + (c \nabla w, v) = 0 && \forall v \in H_0^1(\Omega)^2,\ a.e.\ t \\
&\qquad (\operatorname{div} y, v) = 0 && \forall v \in L^2_{(0)}(\Omega),\ a.e.\ t \\
&(c_t, v) + \frac{1}{Pe}(\nabla w, \nabla v) - (yc, \nabla v) = 0 && \forall v \in H^1(\Omega),\ a.e.\ t \\
&\gamma^2(\nabla c, \nabla(v - c)) - (w, v - c) \geq (c, v - c) && \forall v \in \mathcal{K},\ a.e.\ t
\end{aligned}
$$

$$\mathcal{K} = \{ v \in H^1(\Omega) \,|\, |v| \leq 1\, a.e. \}$$

Numerical handling

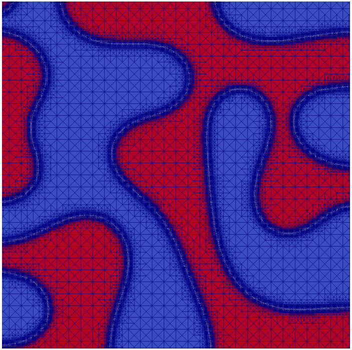
- Variational inequality treated by Moreau-Yosida regularization $\lambda_s(c) = s\max(0, c-1) + s\min(0, c+1)$.
- Semi-implicit time discretization according to Eyre's nonlinear scheme.

$$
\begin{aligned}
\sigma(y - y_{old}, v) + \nu(\nabla y : \nabla v) + ((y_{old} \cdot \nabla)y, v)& \\
+(c\nabla w, v) - (p, \text{div } v) &= 0, \qquad (1)\\
(-\text{div } y, v) &= 0, \qquad (2)\\
(c, v) + \frac{\tau}{Pe}(\nabla w, \nabla v) - \tau(c y_{old}, \nabla v) - (c_{old}, v) &= 0, \qquad (3)\\
\gamma^2(\nabla c, \nabla v) - (w, v) + (\lambda_s(c), v) - (c_{old}, v) &= 0. \qquad (4)
\end{aligned}
$$

$\sigma = 1/\tau$, $\nu = 1/RE$

Remarks

- Navier-Stokes and Cahn-Hilliard are decoupled $\rightarrow$ sequential solving.
- Solution by semi-smooth Newton method for Cahn-Hilliard.
- A-posteriori error estimation used for adaptive spatial discretization $\rightarrow$ talk of Michael Hinze



Instantaneous control of CHNS

AIM: steer $c \to c_d$ using distributed control on the Navier-Stokes equation.

Minimization problem on one time step to obtain control u:

$$\min_u J(c(u), u) = \frac{1}{2}\|c - c_d\|^2 + \frac{\alpha}{2}\|u\|^2 \qquad \text{(P)}$$

s.t.

$$\begin{aligned}
\sigma(y, v) + \nu(\nabla y : \nabla v) & \\
-(p, \operatorname{div}\, v) &= (\sigma y_{old} + u - c_{old}\nabla w_{old} - (y_{old} \cdot \nabla) y_{old}, v), \\
(\operatorname{div}\, y, q) &= 0, \\
(c, v) + \frac{\tau}{Pe}(\nabla w, \nabla v) &= (c_{old}, v) + (c_{old} y, \nabla v), \\
\gamma^2(\nabla c, \nabla v) - (w, v) &= (c_{old} - \lambda_s(c_{old}), v).
\end{aligned}$$

The gradient on J

Abbreviations:

- B the solution operator associated to the quasi-Stokes problem
- $\mathcal{C}$ linear, fourth order solution operator for quasi Cahn-Hilliard problem

Every minimizer fulfills, p_1 adjoint state for c, p_3 adjoint state for y:

$$\begin{aligned}
y &= B\left(\sigma y_{old} - (y_{old}\nabla y_{old})\right) - c_{old}\nabla w_{old} + u\right),\\
c &= \mathcal{C}\left(c_{old} - \tau y \nabla c_{old} + \frac{\tau}{Pe}\Delta(\lambda_s(c_{old}) - c_{old})\right),\\
p_1 &= \mathcal{C}(c - c_d), \text{ and}\\
p_3 &= -\tau B(p_1 \nabla c_{old})\\
\nabla J(c(u), u) &= \alpha u + p_3 = 0.
\end{aligned}$$

Control obtained by IC

Control taken for simulation:
$u = u_0 - \rho \nabla J(u_0) = (1 - \rho\alpha) u_0 - \rho p_3,$
u_0 : initial control.
Here: $u_0 \equiv 0$ and $\rho \equiv 1$

$$\tilde{y} = B(y_{old} - y_{old} \nabla y_{old} - c_{old} \nabla w_{old}),$$
$$\tilde{c} = \mathcal{C}\left(c_{old} - \tau \nabla c_{old} \tilde{y} + \frac{\tau}{Pe} \Delta(\lambda_s(c_{old}) - c_{old}) \right),$$
$$u = \tau B \nabla c_{old} \mathcal{C}\,(\tilde{c} - c_d) =: K(y_{old}, p_{old}, c_{old}, w_{old}).$$

This control is used to perform one step of simulation of the Cahn-Hilliard Navier-Stokes system using the time discretization (1)–(4).

Controlled system

The whole system then is given by

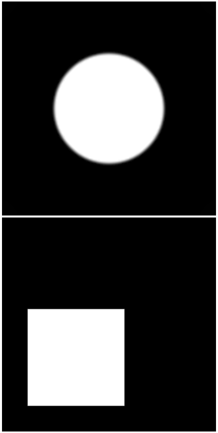
$$\begin{aligned}
\tilde{y} &= B(y_{old} - y_{old}\nabla y_{old} - c_{old}\nabla w_{old}),\\
\tilde{c} &= \mathcal{C}\left(c_{old} - \tau\nabla c_{old}\tilde{y} + \frac{\tau}{Pe}\Delta(\lambda_s(c_{old}) - c_{old})\right),\\
u &= \tau B\nabla c_{old}\mathcal{C}\left(\tilde{c} - c_d\right)
\end{aligned}$$

$$\begin{aligned}
\sigma(y - y_{old}, v) + \nu(\nabla y : \nabla v) + ((y_{old}\cdot\nabla)y, v)&\\
+(c\nabla w, v) - (p, \text{div } v) - (u, v) &= 0,\\
(-\text{div } y, v) &= 0,\\
(c, v) + \frac{\tau}{Pe}(\nabla w, \nabla v) - \tau(c y_{old}, \nabla v) - (c_{old}, v) &= 0,\\
\gamma^2(\nabla c, \nabla v) - (w, v) + (\lambda_s(c), v) - (c_{old}, v) &= 0.
\end{aligned}$$

Outline

Square2Circle I

Morphing a stable distribution (circle) into an unstable (square) and stabilize it



Square2Circle II

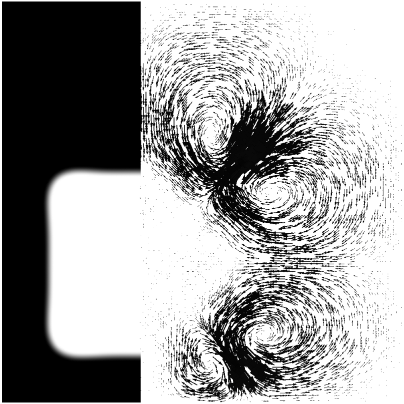


Figure: The stabilized state together with the stabilizing flow.

More complex morphing