Fast reaction limit of a competition-diffusion system

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Purpose : Understand the spatial and temporal behavior of interactive species which are modeled by reaction-diffusion systems

In particular describe their spatial segregation limits in cases where the habitats of the populations become disjoint

We revisit "old problems" and prove some new results

We consider the system

$$(\mathcal{P}^k)\begin{cases} \partial_t u = d_1 \Delta u + f(u) - kuv, & \text{in } \Omega \times \mathbb{R}^+ \\ \partial_t v = d_2 \Delta v + g(v) - \alpha kuv, & \text{in } \Omega \times \mathbb{R}^+ \\ \partial_\nu u = 0, \quad \partial_\nu v = 0, & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u(\cdot, 0) = u_0^k, \quad v(\cdot, 0) = v_0^k, & \text{on } \Omega, \end{cases}$$

where

$f(s) = \lambda s(1-s), g(s) = \mu s(1-s)$;

$k, \alpha, d_1, d_2, \lambda, \mu$ are positive constants ;

$u_0^k, v_0^k \in C(\overline{\Omega}), 0 \le u_0^k, v_0^k \le 1$;

$u_0^k \rightharpoonup u_0,\ v_0^k \rightharpoonup v_0$ in $L^2(\Omega)$ as $k \to \infty$.

u, v are the densities of two biological populations ;

λ, μ are the intraspecific competition rates ;

$k, \alpha k$ are the interspecific competition rates.

References

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• D. Hilhorst, S. Martin, M. Mimura, *Proceedings of the Chiba Conference on Free Boundary Problems* (2008).

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Some main estimates

There exists a unique classical solution (u^k, v^k) of Problem $(\mathcal{P}^k)$. It is such that

$$0 \leq u^k,\ v^k \leq 1 \quad \text{in} \quad Q_T$$

Integrating one of the partial differential equations, we obtain

$$\iint_{Q_T} u^k v^k \leq C(T)/k$$

Prove that $\{u^k\}$ and $\{v^k\}$ are relatively compact in $L^2(Q_T)$. As $k_n \to \infty$,

$u^{k_n} \to u, v^{k_n} \to v$ a.e. in Q_T, and $uv = 0$ a.e. in Q_T.

The limit problem is a free boundary problem. The free boundary separates the regions where $\{u > 0, v = 0\}$ and $\{v > 0, u = 0\}$.

We can give both

- a weak form, where the free boundary does not explicitely appear ;
- a strong form, with explicit boundary conditions.

In order to pass to the limit, we remark that the function

$$w^k = u^k - \frac{v^k}{\alpha}$$

satisfies an equation where k does not appear anymore, namely

$$\begin{cases} \partial_t w^k = d_1 \Delta u^k - \dfrac{d_2}{\alpha} \Delta v^k + f(u^k) - \dfrac{g(v^k)}{\alpha} \\ \partial_\nu w^k = 0. \end{cases}$$

We prove that as $k \to \infty$,

$$u^k \to w^+, \quad v^k \to \alpha w^-,$$

strongly in $L^2(Q_T)$, where the function w is the unique weak solution of the problem

$$(\mathcal{P}) \begin{cases} \partial_t w = \Delta \mathcal{D}(w) + h(w), & in\ \Omega \times \mathbb{R}^+, \\ \partial_\nu w = 0, & on\ \partial\Omega \times \mathbb{R}^+, \\ w(\cdot, 0) = w_0 := u_0 - \dfrac{v_0}{\alpha}, & on\ \Omega, \end{cases}$$

with $\mathcal{D}(s) := d_1 s^+ - d_2 s^-$ and $h(s) := f(s^+) - g(\alpha s^-)$, where $s^+ = max(s, 0)$ and $s^- = -min(s, 0)$.

The strong form of the limit free boundary problem

Assume that, at each time $t \in [0,T]$, there exists a close hypersurface $\Gamma(t)$ and two subdomains $\Omega_u(t)$, $\Omega_v(t)$ such that

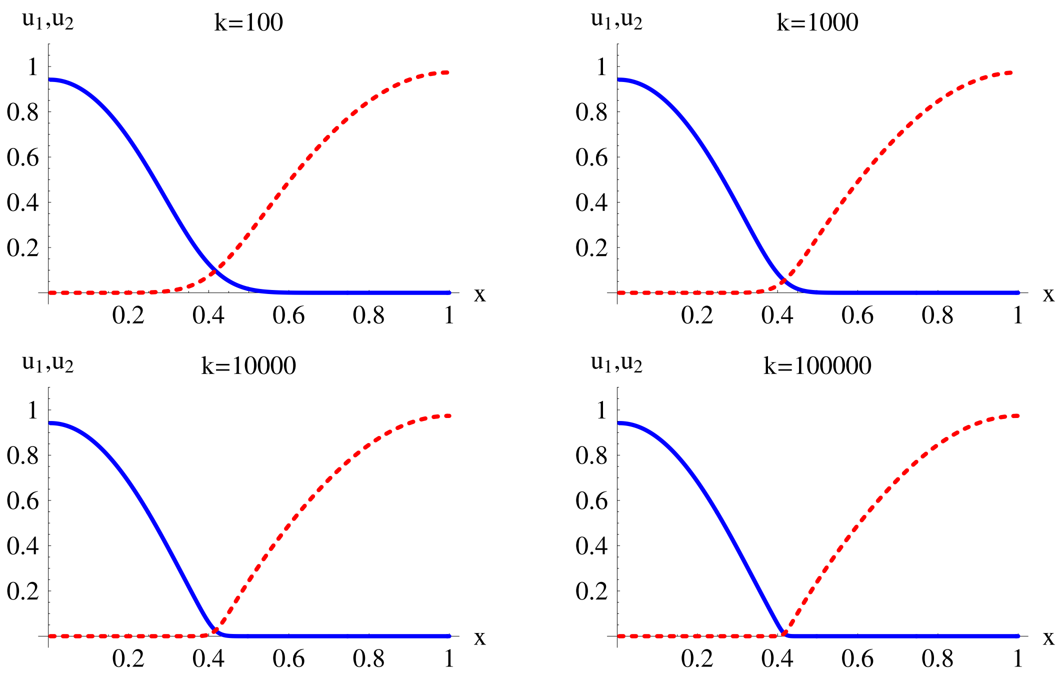
$$\overline{\Omega} = \overline{\Omega_u(t)} \cup \overline{\Omega_v(t)}, \quad \Gamma(t) = \overline{\Omega_u(t)} \cap \overline{\Omega_v(t)},$$

$$w(\cdot,t) > 0 \quad \text{on } \Omega_u(t), \qquad w(\cdot,t) < 0 \quad \text{on } \Omega_v(t).$$

Assume furthermore that $t \mapsto \Gamma(t)$ is smooth enough and that $(u,v) := (w^+, \alpha w^-)$ are smooth up to $\Gamma(t)$. Then the functions u and v satisfy

$$(\mathcal{P}) \begin{cases} \partial_t u = d_1 \Delta u + f(u) & \text{in } \; Q_u := \bigcup \{\Omega_u(t), t \in [0,T]\} \\ \partial_t v = d_2 \Delta v + g(v) & \text{in } \; Q_v := \bigcup \{\Omega_v(t), t \in [0,T]\} \\ u = v = 0 & \text{on } \Gamma := \bigcup \{\Gamma(t), t \in t \in [0,T]\} \\ d_1 \partial_n u = -\dfrac{d_2}{\alpha} \partial_n v & \text{on } \; \Gamma \\ \partial_\nu v = 0 & \text{on } \; \partial\Omega \times [0,T] \\ + \;\; \text{initial conditions.} \end{cases}$$

We should quote a result by Yoshihiro Tonegawa about the regularity of the interface provided that the interface is non degenerate, namely that grad $u \cdot n \neq 0$ on Γ.



u1,u2
k=100
k=1000
k=10000
k=100000
1
0.8
0.6
0.4
0.2
x
0.2
0.4
0.6
0.8
1

More recent questions

Coming back to the problem $(\mathcal{P}^k)$, we have completely characterized the limit (u, v) of (u^k, v^k) as $k \to \infty$.

But what about the singular limit of the term $ku^k v^k$ as $k \to \infty$?

Can we do so both in the case that $d_2 > 0$ and in the case that $d_2 = 0$?

This is joint work with S. Martin and M. Mimura.

THE CASE THAT $d_2 \geq 0$

We only suppose that $d_2 \geq 0$, and consider again Problem $\left(\mathcal{P}^k\right)$:

$$
\left(\mathcal{P}^k\right)\begin{cases}
\partial_t u = d_1 \Delta u + f(u) - kuv, & \text{in } \Omega \times \mathbb{R}^+ \\
\partial_t v = d_2 \Delta v + g(v) - \alpha kuv, & \text{in } \Omega \times \mathbb{R}^+ \\
\partial_\nu u = 0, \quad \partial_\nu v = 0, & \text{on } \partial\Omega \times \mathbb{R}^+ \\
u(\cdot, 0) = u_0^k, \quad v(\cdot, 0) = v_0^k, & \text{on } \Omega,
\end{cases}
$$

Thus if $d_2 > 0$ we have a coupled system of two parabolic equations, whereas a parabolic equation is coupled to an ordinary differential equation if $d_2 = 0$.

D. Hilhorst, M. Röger and J.R. King, *Mathematical Analysis of a model describing the invasion of bacteria in burn wounds*, Nonlinear Anal. 66 (2007), 1118–1140.

D. Hilhorst, M. Röger and J.R. King, *Travelling wave analysis of a model describing tissue degradation by bacteria*, European J. Appl. Math. 18 (2007), 583–605.

We can write it in the single unified form, which we have already seen :

$$(\mathcal{P})\begin{cases} \partial_t w = \Delta\mathcal{D}(w) + h(w), & in\ \Omega \times \mathbb{R}^+, \\ \partial_\nu w = 0, & on\ \partial\Omega \times \mathbb{R}^+, \\ w(\cdot, 0) = w_0 := u_0 - \dfrac{v_0}{\alpha}, & on\ \Omega, \end{cases}$$

with $\mathcal{D}(s) := d_1 s^+ - d_2 s^-$ and $h(s) := f(s^+) - g(\alpha s^-)$, with $d_2 > 0$ or $d_2 = 0$.

Since

$$\int_{Q_T} u^k v^k \leq C(T)/k \qquad ,$$

it follows that there exists a measure μ such that $ku^kv^k \rightharpoonup \mu$ as $k \to \infty$, in the sense of the weak convergence of measures.

We only compute μ in the case that the limit problem can be written in a strong form with a smooth interface. More precisely, we prove the following result.

Theorem. *There exists a measure μ such that*

$$k \quad u^k \quad v^k \rightharpoonup \mu, \quad \textit{in the sense of measures.}$$

If the interface Γ is smooth, then μ is localized on Γ and is given by

$$\mu(x,t) = \frac{1}{1+\alpha}\left([d_1\,\partial_n u + d_2\,\partial_n v] + [v]\ V_n\right)\delta(x - \xi(t)),$$

where the function $t \mapsto \xi(t)$ is a parametrization of the free boundary Γ.

Remark. *An alternative expression for the measure μ is given by*

$$\mu(x,t) = \begin{cases} \frac{1}{1+\alpha}\left[(d_1\partial_n u + d_2\partial_n v)\,(\xi(t),t)\right]\,\delta(x-\xi(t)), & \text{if } d_2 > 0, \\ \frac{1}{\alpha}\left[v(\xi(t),t)\right]\,V_n\,\delta(x-\xi(t)), & \text{if } d_2 = 0. \end{cases}$$

Defining $\mu^k = k\,F(u^k, v^k)$ and using $\psi \in C_0^\infty(Q_T)$, we have :

$$\iint_{Q_T} \mu^k\,\psi = \iint_{Q_T} \Big(u^k\,\partial_t\psi + d_1\,u^k\,\Delta\psi + f(u^k)\psi\Big) = \frac{1}{\alpha}\iint_{Q_T} \Big(v^k\,\partial_t\psi + d_2\,v^k\,\Delta\psi + g(v^k)\psi\Big)\,.$$

Therefore, letting $k \to \infty$ gives

$$\iint_{Q_T} \mu\,\psi = \iint_{Q_T} (u\,\partial_t\psi + d_1\,u\,\Delta\psi + f(u)\psi) = \frac{1}{\alpha}\iint_{Q_T} (v\,\partial_t\psi + d_2\,v\,\Delta\psi + g(v)\psi)$$

which we integrate by parts to obtain

$$
\begin{aligned}
\iint_{Q_T} \mu\,\psi \quad &= \int_0^T \int_{\Omega_u(t)} \underbrace{(-\partial_t u + d_1 \Delta u + f(u))}_{=0} \psi + \int_0^T \int_{\Gamma(t)} \underbrace{[u]}_{=0} (V_n\,\psi - d_1 \partial_n \psi) + [d_1 \partial_n u]\,\psi \\
\alpha \iint_{Q_T} \mu\,\psi \quad &= \int_0^T \int_{\Omega_v(t)} \underbrace{(-\partial_t v + d_2\,\Delta v + g(v))}_{=0} \psi + \int_0^T \int_{\Gamma(t)} [v] V_n\,\psi - \underbrace{d_2 [v]}_{=0} \partial_n \psi + [d_2 \partial_n v]\,\psi.
\end{aligned}
$$

This yields

$$
\iint_{Q_T} \mu\,\psi = \frac{1}{1+\alpha} \int_0^T \int_{\Gamma(t)} \left([d_1\,\partial_n u + d_2\,\partial_n v] + [v]\ V_n \right)\ \psi.
$$

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$$
(\mathrm{P}^\lambda) \qquad \begin{cases}
\dfrac{\partial u}{\partial t} = \Delta u - \alpha\lambda G(u,v,w) & \text{in} \quad Q_T := \Omega \times (0,T), \\
\dfrac{\partial v}{\partial t} = \Delta v - \beta\lambda G(u,v,w) & \text{in} \quad Q_T, \\
\dfrac{\partial w}{\partial t} = \lambda G(u,v,w) & \text{in} \quad Q_T, \\
\dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,T), \\
u(\cdot,0) = u_0, \ v(\cdot,0) = v_0, \ w(\cdot,0) = w_0 & \text{in} \quad \Omega,
\end{cases}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, T, λ, α are positive constants and β is a positive or negative constant, and n is the outward normal unit vector to the boundary.

Problem (P^λ) models reactive transport in a cement based material where one mineral species and two aqueous species react according to a kinetic law.

w : concentration of a mineral,
u and v : concentrations of the species in the liquid phase.

The function G is given by

$$G(u,v,w) = F(u,v)^+ - \mathrm{sgn}^+(w)F(u,v)^-,$$

where,

$$s^+ = \max(0,s), \quad s^- = \max(0,-s), \quad \mathrm{sgn}(s) = \begin{cases} 1 & \text{if} \quad s>0, \\ 0 & \text{if} \quad s=0, \\ -1 & \text{if} \quad s<0. \end{cases}$$

The reaction term F is of the form

$$F(u,v) = u^{\alpha} v^{\beta^+} - K v^{\beta^-},$$

where K is a positive constant, and α and β are algebraic stoichiometric coefficients, and thus integers.

A technical problem is that the reaction function G is not sign-definite.

In the special case that $F(u,v) = u - \overline{u}$, Problem (P^{λ}) reduces to a system of two equations for u and w. The singular limit of the solution $(u^{\lambda}, w^{\lambda})$ as $\lambda \to \infty$ has been studied by Bouillard, Eymard, Henry, Herbin and Hilhorst, Nonlinear Analysis, Real World Applications 2009.

Set

$$h = v - \frac{\beta}{\alpha} u.$$

and remark that h satisfies the heat equation together with homogeneous Neumann boundary conditions and the initial condition $h(0) = v_0 - \frac{\beta}{\alpha} u_0$. We suppose that

(H1) There exists a positive constant M such that

$$0 \leq u_0, v_0, w_0 \leq M.$$

(H2)
- if $\beta > 0$, then for all $u, v \geq 0$, $F(u, v)$ is increasing with u and v, and the inequalities $F(0, v) \leq 0$, $F(u, 0) \leq 0$ and $F(0, 0) \leq 0$ hold,
- if $\beta = 0$, then F is independent of the second variable, and for all $u \geq 0$, $F(u, 0)$ is increasing with u, and the inequality $F(0, 0) \leq 0$ holds,
- if $\beta < 0$, then for all $u, v \geq 0$, $F(u, v)$ is increasing with u and decreasing with v, and the inequalities $F(0, v) \leq 0$, $F(u, 0) \geq 0$ hold.

Conditions on F

(H3) There exist a non-negative function f depending only on h, α, β and K and a positive function g depending only on u, h, α, β, K such that f is smooth enough, satisfies a homogeneous Neumann boundary condition and

$$F\left(u, h(x,t) + \frac{\beta}{\alpha} u\right) = (u - f(x,t))g(x,t)$$

for all $u \geq 0$ and a.e. $(x,t) \in Q_T$. Examples are given by

1. Case $\alpha \in N$, $\beta = 0$, $F(u,v) = u^\alpha - K$,

$$f = K^{1/\alpha} > 0, \quad g = \sum_{k=1}^{\alpha} u^{\alpha-k} K^{\frac{k-1}{\alpha}} \geq K^{\frac{\alpha-1}{\alpha}} > 0,$$

2. Case $\alpha = 1$, $\beta = -1$, $F(u,v) = u - Kv$,

$$f = \frac{K}{1+K} h \geq 0, \quad g = 1 + K > 0.$$

Theorem. As λ tends to infinity, the solution $(u^\lambda, v^\lambda, w^\lambda)$ of Problem (P^λ) is such that

$$u^\lambda \to f - Z^+, \quad v^\lambda \to h + \frac{\beta}{\alpha} f - \frac{\beta}{\alpha} Z^+, \quad w^\lambda \to \frac{1}{\alpha} Z^-,$$

where Z is the unique weak solution of the Stefan problem

$$\text{(SP)} \qquad \begin{cases} \dfrac{\partial Z}{\partial t} = \Delta Z^+ + \dfrac{\partial f}{\partial t} - \Delta f & \text{in} \quad Q_T, \\ \dfrac{\partial Z^+}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ Z(\cdot, 0) = f(0) - u_0 - \alpha w_0 & \text{in} \quad \Omega, \end{cases}$$

Preliminary lemma

There exists a positive constant $C(\lambda)$ depending on T, α, β, λ, F, u_0, v_0, w_0 such that

$$0 \leq u^\lambda, v^\lambda, w^\lambda \leq C(\lambda) \quad \textit{in } Q_T.$$

Lemma

(i) u^λ and v^λ are uniformly bounded in $L^\infty(Q_T)$ with respect to λ.

(ii) Suppose that either $\alpha \in \mathbb{N}$, $\beta = 0$ or $\alpha = 1$, $\beta = -1$. Then, w^λ is uniformly bounded in $L^\infty(Q_T)$ with respect to λ.

We define

$$z^\lambda := f - u^\lambda - \alpha w^\lambda$$

.

Lemma

$$(i)\ \left\|G(u^\lambda, v^\lambda, w^\lambda)\right\|_{L^1(Q_T)} \leq C/\lambda,$$

$$(ii)\ \left\|w^\lambda\right\|_{W^{1,1}(0,T;L^1(\Omega))} \leq C,$$

$$(iii)\ \left\|u^\lambda\right\|_{L^2(0,T;H^1(\Omega))} + \left\|v^\lambda\right\|_{L^2(0,T;H^1(\Omega))} + \left\|z^\lambda\right\|_{H^1(0,T;H^1(\Omega)')} \leq C.$$

Theorem Let $(u^\lambda, v^\lambda, w^\lambda)$ be the weak solution of (P^λ) and Z the weak solution of (SP). Set $z^\lambda = f - u^\lambda - \alpha w^\lambda$. If there is a positive constant M independent of λ such that

$$\|w^\lambda\|_{L^\infty(Q_T)} \leq M,$$

then there exists a positive constant C independent of λ such that

$$\begin{aligned}\left\|u^\lambda - (f - Z^+)\right\|_{L^2(Q_T)} + \left\|v^\lambda - \left(h + \frac{\beta}{\alpha} f - \frac{\beta}{\alpha} Z^+\right)\right\|_{L^2(Q_T)} \\ + \left\|z^\lambda - Z\right\|_{L^\infty(0,T;H^1(\Omega)^*)} \leq C\lambda^{-1/2}.\end{aligned}$$

I thank you for your attention