Derivation of entropy inequalities at interfaces via asymptotic expansions

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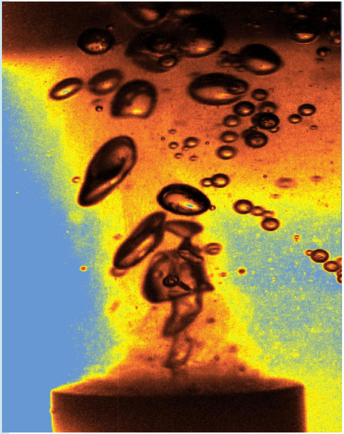
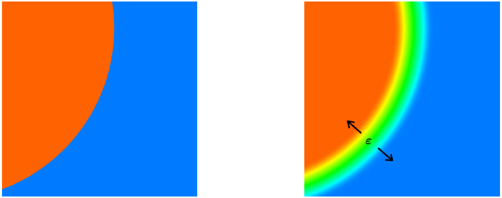
Motivation



- model single substance flow
- two phases (liquid, vapor)
- isothermal (for simplicity)
- compressible
- including phase transitions (due to pressure changes)

Sharp interface limit



sharp interface models   diffuse interface models

Compute the sharp interface limit of a diffuse interface model:

- justify the diffuse interface model,
- relate its parameters to macroscopic quantities,
- motivate a kinetic relation for the sharp interface model,
- what is the entropy dissipation, how is it related to the kinetic relation?

Outline

1. Introduction to the Navier-Stokes-Korteweg system
2. General sharp interface models
3. A sharp interface limit
4. The entropy inequality for the SI limit
5. Prospects

The NSK Equations

The local Navier-Stokes-Korteweg Model:

(Dunn&Serrin '85)
$\rho(\mathbf{x}, t) > 0$ density, $\mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^d$ velocity, $p(\rho) > 0$ pressure given by constitutive relation.

$$\left.\begin{aligned} \rho_t + \quad \operatorname{div}(\rho\mathbf{u}) &= 0 \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u} + p(\rho)\mathcal{I}) &= \operatorname{div}(\sigma_{\mathsf{NS}}) + \gamma\rho\nabla\Delta\rho \end{aligned}\right\} \text{ in } D \times \mathbb{R}_{>0},$$

$$\sigma_{\mathsf{NS}} = \lambda(\operatorname{div}\mathbf{u})\mathcal{I} + \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T),$$

$$\mathbf{u} = 0, \quad \nabla\rho\cdot\mathbf{n} = 0 \qquad \text{in } \partial D \times \mathbb{R}_{>0}.$$

$$\sigma_{\mathsf{K}} := \left(\rho\Delta\rho + \frac{1}{2}|\nabla\rho|^2\right)\mathcal{I} - \nabla\rho\otimes\nabla\rho$$

$$\operatorname{div}(\sigma_{\mathsf{K}}) = \rho\nabla\Delta\rho.$$

The NSK Equations

The local Navier-Stokes-Korteweg Model:

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$$\mathbf{u} = 0, \quad \nabla\rho\cdot\mathbf{n} = 0 \qquad \text{in } \partial D \times \mathbb{R}_{>0}.$$

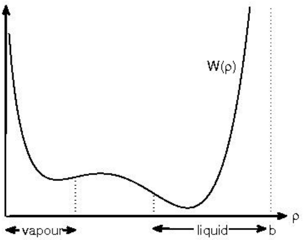
Energy/Entropy equality:

$$\left(W(\rho) + \frac{\rho}{2}|\mathbf{u}|^2 + \frac{\gamma}{2}|\nabla\rho|^2\right)_t + \operatorname{div}\left(\left(W(\rho) + \frac{\rho}{2}|\mathbf{u}|^2 + \frac{\gamma}{2}|\nabla\rho|^2\right)\mathbf{u}\right)$$
$$+ \operatorname{div}\left((p(\rho) - \sigma_{\mathsf{NS}} - \sigma_{\mathsf{K}})\,\mathbf{u} + \gamma\rho\nabla\rho(\nabla\cdot\mathbf{u})\right)$$
$$= -\sigma_{\mathsf{NS}} : (\nabla\mathbf{u}) \leqslant 0.$$

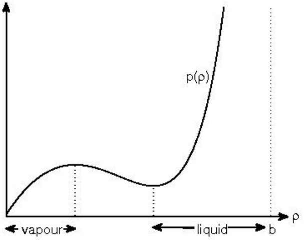
Van der Waals pressure and energy

To have two phases, we need a non-monotone pressure function.

Helmholtz energy density



pressure



$$p(\rho) = \rho W'(\rho) - W(\rho), \qquad p'(\rho) = \rho W''(\rho).$$

The first order part is hyperbolic provided $p'(\rho) > 0$.
$\Longrightarrow$ Problem of hyperbolic-elliptic type.

Sharp interface framework: In the bulk

A set of PDEs in each bulk domain, e.g.

isothermal Euler equations

$$\begin{aligned} \rho_t + \operatorname{div}(\rho \mathbf{u}) &= 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) &= 0. \end{aligned}$$

Smooth solutions satisfy the entropy equality

in the bulk

$$\left(W(\rho) + \frac{\rho}{2} |\mathbf{u}|^2 \right)_t + \operatorname{div} \left(\left(W(\rho) + \frac{\rho}{2} |\mathbf{u}|^2 + p(\rho) \right) \mathbf{u} \right) = 0.$$

Sharp interface framework: At the interface

Conservation/ balance at the interface is equivalent to

Rankine-Hugoniot conditions

$$\begin{aligned} [\![\rho(\mathbf{u}\cdot\boldsymbol{\nu} - w_\nu)]\!] &= 0, \\ [\![\rho\mathbf{u}(\mathbf{u}\cdot\boldsymbol{\nu} - w_\nu) + \boldsymbol{\nu} p(\rho)]\!] &= \boldsymbol{\nu}\sigma\kappa, \end{aligned}$$

- $\boldsymbol{\nu}$ unit normal vector to the interface,
- w_ν normal velocity of the interface,
- κ is the sum of the principal curvatures, σ surface tension.

Entropy inequality

$$\left[\!\!\left[\rho(\mathbf{u}\cdot\boldsymbol{\nu} - w_\nu)\left(W'(\rho) + \frac{1}{2}|\mathbf{u} - \mathbf{w}|^2\right)\right]\!\!\right] \leqslant 0.$$

Uniqueness of solutions

Rankine Hugoniot conditions + entropy inequality $\not\Rightarrow$ uniqueness.
Overview on well-posedness in 1D, see LeFloch, Hyperbolic systems of conservation laws.

We need an additional condition called **kinetic relation**,

$$\varphi(\rho^-, \rho^+, \mathbf{u}^-, \mathbf{u}^+, \mathbf{w}) = 0.$$

It must be compatible with the Entropy inequality.

Theorem (Benzoni-Gavage, Freistühler '04):

The free boundary value problem for the Euler equations with a van-der-Waals pressure function is locally well-posed, provided one imposes the Rankine-Hugoniot conditions and zero entropy dissipation at the interface, i.e.

$$\left[\!\left[W'(\rho) + \frac{1}{2}|\mathbf{u} - \mathbf{w}|^2 \right]\!\right] = 0.$$

Jump conditions including surface quantities

Surface quantities lead to more general jump conditions

Satz: Dreyer '03

$$[\![\underbrace{\rho(u_\nu - w_\nu)}_{=:j}]\!] = 0,$$

$$[\![\rho(u_\nu - w_\nu)(\mathbf{u} - \mathbf{w}) + \boldsymbol{\nu} p(\rho)\ ]\!] = 0,$$

and satisfy

$$+\,\{j\}\left[\!\!\left[W'(\rho) + \frac{|\mathbf{u} - \mathbf{w}|^2}{2}\right]\!\!\right] = 0,$$

where

$$[\![a]\!] := a^+ - a^-, \qquad \{a\} := \frac{a^+ + a^-}{2},$$

- u_ν normal velocity of the fluid,

Jump conditions including surface quantities

Surface quantities lead to more general jump conditions

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$$\underbrace{[\![\rho(u_\nu - w_\nu)]\!]}_{=:j} = -\frac{\partial \rho_\Gamma}{\partial t} - \rho_\Gamma \left(\mathrm{div}_\Gamma(w_\theta) - \kappa w_\nu\right),$$

$$[\![\rho(u_\nu - w_\nu)(\mathbf{u} - \mathbf{w}) + \boldsymbol{\nu} p(\rho)]\!] = 0,$$

and satisfy

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where

$$[\![a]\!] := a^+ - a^-, \qquad \{a\} := \frac{a^+ + a^-}{2},$$

- ρ_Γ surface mass density, div_Γ surface divergence,
- w_θ tangential velocity of the interface.

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$$[\![\rho(u_\nu - w_\nu)(\mathbf{u} - \mathbf{w}) + \boldsymbol{\nu} p(\rho)\ ]\!] = -\frac{\partial \mathbf{w}}{\partial t}\rho_\Gamma + \mathrm{div}_\Gamma(\boldsymbol{\sigma}_\Gamma),$$

and satisfy

$$+\{j\}\left[\!\!\left[W'(\rho) + \frac{|\mathbf{u} - \mathbf{w}|^2}{2}\right]\!\!\right] = 0,$$

where

$$[\![a]\!] := a^+ - a^-, \qquad \{a\} := \frac{a^+ + a^-}{2},$$

- ρ_Γ surface mass density,
- $\boldsymbol{\sigma}_\Gamma$ the surface stress vector.

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$$[\![\rho(u_\nu - w_\nu)(\mathbf{u}-\mathbf{w}) + \boldsymbol{\nu} p(\rho)\ ]\!] = -\frac{\partial \mathbf{w}}{\partial t}\rho_\Gamma + \mathrm{div}_\Gamma(\boldsymbol{\sigma}_\Gamma),$$

and satisfy

$$\frac{\partial W_\Gamma}{\partial t} - (\gamma_\Gamma - W_\Gamma)\left(\mathrm{div}_\Gamma(w_\theta) - \kappa w_\nu\right)$$
$$+[\![j]\!]\left\{W'(\rho) + \frac{|\mathbf{u}-\mathbf{w}|^2}{2}\right\} + \{j\}\left[\!\!\left[W'(\rho) + \frac{|\mathbf{u}-\mathbf{w}|^2}{2}\right]\!\!\right] \leqslant 0,$$

- W_Γ surface Helmholtz free energy density,
- γ_Γ surface tension, given by $\boldsymbol{\sigma}_\Gamma = \gamma_\Gamma \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|^2}$.

Aim of SI limit

- Derive SI limit fitting into this framework, i.e, determine conditions for

$$[\![\rho(u_\nu - w_\nu)]\!],$$
$$[\![\rho(u_\nu - w_\nu)(\mathbf{u}-\mathbf{w}) + \boldsymbol{\nu} p(\rho)]\!],$$
$$\left[\!\!\left[W'(\rho) + \frac{|\mathbf{u}-\mathbf{w}|^2}{2} \right]\!\!\right]$$

  and determine parameters ρ_Γ, $\boldsymbol{\sigma}_\Gamma$, γ_Γ, W_Γ. Surface quantities given in terms of the solutions to the "inner equation".
- These jump conditions determine the energy dissipation.
- SI -entropy inequality can be directly derived from the "continuous" entropy inequality.

Choose a scaling

We non-dimensionalise the equations and choose

$$M = \mathcal{O}(1), \quad \text{Re} := \mathcal{O}(\varepsilon^{-2}), \quad \frac{t_r^2 \rho_r}{x_r^4}\gamma_r = \mathcal{O}(\varepsilon^2).$$

Scaled version of the NSK system

$$\begin{aligned} \rho_t + \nabla \cdot (\rho \mathbf{u}) &= 0, \\ (\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) &= \varepsilon^2 \text{div}(\mathbf{S}) + \gamma \varepsilon^2 \rho \nabla \Delta \rho, \end{aligned}$$

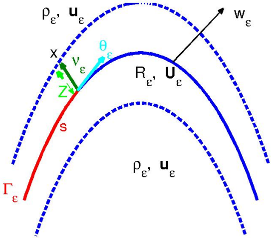
which means that the magnitudes of viscosity and capillarity are of the same (small) order.
For a low Mach number scaling, see Hermsdörfer, Kraus, Kröner '09.

Decomposition, and coordiante change at the interface

Decomposition the problem into

- "outer problem" away from the interface,
- "inner problem" inside the interfacial layer,

these are linked by "matching conditions"



New coordinates (z, s, τ) in the interfacial layer

$$(\mathbf{x}, t) = (\mathbf{r}_\varepsilon(s, \tau) + \varepsilon z \boldsymbol{\nu}_\varepsilon(s, \tau), \tau),$$

where $\mathbf{r}_\varepsilon(\cdot, t)$ is a parametrization of the interface

$$\Gamma_\varepsilon(t) := \left\{ \mathbf{x} \in \mathbb{R}^2 : \rho_\varepsilon(\mathbf{x}, t) = \rho_* \right\},$$

where $\rho_* \in (0, b)$ such that $p'(\rho_*) < 0$.

Assumptions

Quantities in inner coordinates (denoted by capital letters):

$$R_\varepsilon(\tau, s, z) = \sum_{i=0}^{\infty} \varepsilon^i R_i(\tau, s, z) \quad \text{and} \quad \mathbf{U}_\varepsilon(\tau, s, z) = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{U}_i(\tau, s, z).$$

Quantities in outer coordinates

$$\rho_\varepsilon(\mathbf{x}, t) = \sum_{i=0}^{\infty} \varepsilon^i \rho_i(\mathbf{x}, t) \quad \text{and} \quad \mathbf{u}_\varepsilon(\mathbf{x}, t) = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{u}_i(\mathbf{x}, t).$$

Position of the interface $\Gamma_\varepsilon(t) := \{\mathbf{x} \in \mathbb{R}^2 : \rho_\varepsilon(\mathbf{x}, t) = \rho_*\}$

$$r_\varepsilon(\tau, s) = \sum_{i=0}^{\infty} \varepsilon^i r_i(\tau, s).$$

Inner equations: leading order

We insert the inner expansions into (NSK) and change the coordinates. Collecting the terms of order ε^{-1} yields

$$\begin{aligned} -w_\nu R_{0,z} + (R_0 \boldsymbol{\nu}_0 \cdot \mathbf{U}_0)_z &= 0, \\ ((\boldsymbol{\nu}_0 \cdot \mathbf{U}_0 - w_\nu)\boldsymbol{\nu}_0 \cdot \mathbf{U}_0)_z + W'(R_0)_z &= \gamma R_{0,zzz}. \end{aligned} \tag{IE}$$

The first equation implies that the zeroth order mass flux

$$j_0 := R_0(\boldsymbol{\nu}_0 \cdot \mathbf{U}_0 - w_{\nu 0})$$

is constant with respect to z. Hence,

$$\Longrightarrow [\![\rho_0(u_{\nu 0} - w_{\nu 0})]\!] = 0.$$

Inner equations: leading order

Theorem (Benzoni-Gavage, Danchin, Descombes, Jamet, '07):

For $|j_0| \ll 1$ there exist $\rho_0^\pm(j_0) > 0$ such that

$$\begin{aligned} \left[\!\left[W'(\rho_0) + \frac{1}{2}\frac{j_0^2}{(\rho_0)^2} \right]\!\right] &= 0, \\ \left[\!\left[p(\rho_0) + \frac{j_0^2}{\rho_0} \right]\!\right] &= 0. \end{aligned}$$

Furthermore there exists a solution $R_0(j_0)$ of (IE) satisfying

$$R_0(j_0) \overset{z\to\pm\infty}{\longrightarrow} \rho_0^\pm(j_0).$$

The interfacial normal velocity $w_{\nu 0}$ can be computed from mass flux and density.

Inner equations: leading order

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The interfacial normal velocity $w_{\nu 0}$ can be computed from mass flux and density.

Inner equations:

For $j_0 \neq 0$ (IE) also implies

$$[\![\boldsymbol{\theta}_0 \cdot \mathbf{u}_0]\!] = 0.$$

So we can choose the parameterisation of the interface such that

$$w_\theta = u_{\theta 0}.$$

The $\mathcal{O}(\varepsilon)$ order of the inner equations yields an inhomogeneous, linear ODE system for $R_1, \mathbf{U}_1$.

By the Fredholm Alternative we find solvability conditions, which yield the $\mathcal{O}(\varepsilon)$ order of the jump conditions.

Jump Conditions up to $\mathcal{O}(\varepsilon)$ I

Theorem (Dreyer, Giesselmann, Kraus, Rohde, '10):

Under standard assumptions for asymptotic analysis the outer quantities are subject to the following interface conditions

$$[\![\rho_\varepsilon((\mathbf{u}_\nu)_\varepsilon - (w_\nu)_\varepsilon)]\!] = -\frac{\partial \rho_\Gamma}{\partial \tau} - \rho_\Gamma \left(\mathrm{div}_\Gamma(w_{\theta 0}) - \kappa_0 w_{\nu 0}\right) + \mathcal{O}(\varepsilon^2),$$

where ρ_Γ is the mass attributed to the interface and div_Γ is the surface divergence:

$$\begin{aligned} \rho_\Gamma &:= \varepsilon \int_0^\infty R_0 - \rho_0^+ \, dz + \varepsilon \int_{-\infty}^0 R_0 - \rho_0^- \, dz, \\ \mathrm{div}_\Gamma(w_{\theta 0}) &:= \frac{1}{\|\boldsymbol{\theta}_0\|} \left(\|\boldsymbol{\theta}_0\| w_{\theta 0}\right)_s . \end{aligned}$$

Jump Conditions up to $\mathcal{O}(\varepsilon)$ II

Theorem: continued

$$[\![\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)(\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon) + \boldsymbol{\nu}_\varepsilon p(\rho_\varepsilon)]\!] = -\frac{\partial \mathbf{w}_0}{\partial \tau}\rho_\Gamma + \mathrm{div}_\Gamma(\boldsymbol{\sigma}_\Gamma) + \mathcal{O}(\varepsilon^2),$$

where $\boldsymbol{\sigma}_\Gamma$ is the surface stress vector given by $\sigma_\Gamma^j = \gamma_\Gamma \frac{\theta_0^j}{\|\boldsymbol{\theta}_0\|^2}$ with

$$\gamma_\Gamma = \varepsilon \int_0^\infty \left(\frac{j_0^2}{R_0} - \frac{j_0^2}{\rho_0^+} + \gamma R_{0,z}^2 \right) dz + \varepsilon \int_{-\infty}^0 \left(\frac{j_0^2}{R_0} - \frac{j_0^2}{\rho_0^-} + \gamma R_{0,z}^2 \right) dz.$$

When $(\gamma_\Gamma)_s = 0 \Longrightarrow$ Young-Laplace like law

Jump Conditions up to $\mathcal{O}(\varepsilon)$ III

Theorem: continued

$$\left[\!\!\left[\frac{1}{2}|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2 + W'(\rho_\varepsilon)\right]\!\!\right] = -\varepsilon \int_0^\infty \boldsymbol{\nu}_0 \cdot (\mathbf{U}_0 - \mathbf{u}_0^+)_\tau \, dz$$
$$-\varepsilon \int_{-\infty}^0 \boldsymbol{\nu}_0 \cdot (\mathbf{U}_0 - \mathbf{u}_0^-)_\tau \, dz$$
$$-\varepsilon \left(\lambda + 2\mu\right) j_0 \int_{-\infty}^\infty \left(\left(\frac{1}{R_0}\right)_z\right)^2 dz$$
$$+\mathcal{O}(\varepsilon^2).$$

λ, μ are bulk and shear viscosity parameters.

Jump Conditions up to $\mathcal{O}(\varepsilon)$ III

Theorem: continued

$$\left[\!\left[\frac{1}{2}|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2 + W'(\rho_\varepsilon) \right]\!\right] = -\varepsilon \int_0^\infty \boldsymbol{\nu}_0 \cdot (\mathbf{U}_0 - \mathbf{u}_0^+)_\tau \, dz - \varepsilon \int_{-\infty}^0 \boldsymbol{\nu}_0 \cdot (\mathbf{U}_0 - \mathbf{u}_0^-)_\tau \, dz - \varepsilon (\lambda + 2\mu) j_0 \int_{-\infty}^\infty \left(\left(\frac{1}{R_0} \right)_z \right)^2 dz + \mathcal{O}(\varepsilon^2).$$

λ, μ are bulk and shear viscosity parameters.

Is this a reasonable condition?

Entropy dissipation at the interface

Theorem (Dreyer, Giesselmann, Kraus, Rohde, '10):

For ε sufficiently small the above jump conditions imply

$$\begin{aligned}
0 \;&\geqslant\; -\varepsilon \overbrace{(\lambda + 2\mu)}^{>0} j_0^2 \int_{-\infty}^{\infty} \left(\left(\frac{1}{R_0} \right)_z \right)^2 dz + \mathcal{O}(\varepsilon^2) \\
&= \frac{\partial W_\Gamma}{\partial \tau} - (\gamma_\Gamma - W_\Gamma)\left(\mathrm{div}_\Gamma((w_\theta)_\varepsilon) - \kappa_\varepsilon (w_\nu)_\varepsilon\right) \\
&\quad + [\![j_\varepsilon]\!] \left\{ W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2} \right\} \\
&\quad + \{j_\varepsilon\} \left[\!\!\left[W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2} \right]\!\!\right],
\end{aligned}$$

the jump conditions are compatible with the 2nd law of Thermodynamics.

Entropy dissipation at the interface

Theorem: continued

$$W_\Gamma = \varepsilon \int_0^\infty \left(W(R_0) - W(\rho_0^+) + \frac{1}{2}\frac{j_0^2}{R_0} - \frac{1}{2}\frac{j_0^2}{\rho_0^+} + \frac{\gamma}{2} R_{0,z}^2 \right) dz$$
$$+ \varepsilon \int_{-\infty}^0 \left(W(R_0) - W(\rho_0^-) + \frac{1}{2}\frac{j_0^2}{R_0} - \frac{1}{2}\frac{j_0^2}{\rho_0^-} + \frac{\gamma}{2} R_{0,z}^2 \right) dz,$$

$$\gamma_\Gamma = \varepsilon \int_0^\infty \left(\frac{j_0^2}{R_0} - \frac{j_0^2}{\rho_0^+} + \gamma R_{0,z}^2 \right) dz + \varepsilon \int_{-\infty}^0 \left(\frac{j_0^2}{R_0} - \frac{j_0^2}{\rho_0^-} + \gamma R_{0,z}^2 \right) dz$$

Gibbs adsorption law

A straightforward computation shows

$$W_\Gamma - \gamma_\Gamma = \rho_\Gamma \left(g(\rho_0^\pm) + \frac{1}{2} \left(\frac{j_0}{\rho_0^\pm} \right)^2 \right)$$

which is a special case of the Gibbs adsorption law.

A different way to determine energy dissipation:

Apply coordinate change and inner expansion to the scaled entropy equality:

$$\left(W(\rho)+\frac{\rho}{2}|\mathbf{u}|^2+\frac{\gamma\varepsilon^2}{2}|\nabla\rho|^2\right)_t+\mathsf{div}\left(\left(W(\rho)+\frac{\rho}{2}|\mathbf{u}|^2+\frac{\gamma\varepsilon^2}{2}|\nabla\rho|^2\right)\mathbf{u}\right)$$
$$+\,\mathrm{div}\left(\left(p(\rho)-\varepsilon^2\sigma_{\mathsf{NS}}-\varepsilon^2\sigma_{\mathsf{K}}\right)\mathbf{u}+\varepsilon^2\gamma\rho\nabla\rho(\nabla\cdot\mathbf{u})\right)$$
$$=-\varepsilon^2\sigma_{\mathsf{NS}}:(\nabla\mathbf{u}).$$

Then gathering $\mathcal{O}(\varepsilon^{-1})$ terms gives

$$\begin{aligned}0=&-w_{\nu 0}\left(W(R_0)+\frac{R_0}{2}|\mathbf{U}_0|^2+\frac{\gamma}{2}(R_{0,z})^2\right)_z\\&+\nu_0^i\left(\left(W(R_0)+\frac{R_0}{2}|\mathbf{U}_0|^2+\frac{\gamma}{2}(R_{0,z})^2\right)U_0^i\right)_z\\&+\left(p(R_0)\nu_0^iU_0^i-\gamma(R_{0,zz}R_0-\frac{1}{2}(R_{0,z})^2)\nu_0^iU_0^i+\gamma R_0R_{0,z}\nu^iU_{0,z}^i\right)_z.\end{aligned}$$

A different way to determine energy dissipation:

This equation can directly be integrated and due to the matching conditions we obtain

$$\left[\!\left[\left(W(\rho_0) + \frac{\rho_0}{2}|\mathbf{u}_0|^2 \right)(u_{\nu 0} - w_{\nu 0}) + p(\rho_0) u_{\nu 0} \right]\!\right] = 0$$

A different way to determine energy dissipation:

This equation can directly be integrated and due to the matching conditions we obtain

$$\left[\!\left[\left(W(\rho_0) + \frac{\rho_0}{2}|\mathbf{u}_0|^2 \right) (u_{\nu 0} - w_{\nu 0}) + p(\rho_0) u_{\nu 0} \right]\!\right] = 0$$

$$\left[\!\left[\left(W(\rho_0) + p(\rho_0) + \frac{\rho_0}{2}|\mathbf{u}_0 - \mathbf{w}_0|^2 \right) (u_{\nu 0} - w_{\nu 0}) \right]\!\right] = 0$$

Use $[\rho_0(u_{\nu 0} - w_{\nu 0})] = 0$, $[\rho_0(u_{\nu 0} - w_{\nu 0})^2 + p(\rho_0)] = 0$ and $u_{\theta 0} = w_{\theta 0}$.

A different way to determine energy dissipation:

This equation can directly be integrated and due to the matching conditions we obtain

$$\left[\!\left[\left(W(\rho_0) + \frac{\rho_0}{2}|\mathbf{u}_0|^2 \right)(u_{\nu 0} - w_{\nu 0}) + p(\rho_0)\, u_{\nu 0} \right]\!\right] = 0$$

$$\left[\!\left[\left(W(\rho_0) + p(\rho_0) + \frac{\rho_0}{2}|\mathbf{u}_0 - \mathbf{w}_0|^2 \right)(u_{\nu 0} - w_{\nu 0}) \right]\!\right] = 0$$

$$\left[\!\left[\left(\rho_0\, W'(\rho_0) + \frac{\rho_0}{2}|\mathbf{u}_0 - \mathbf{w}_0|^2 \right)(u_{\nu 0} - w_{\nu 0}) \right]\!\right] = 0$$

Use $W(\rho_0) + p(\rho_0) = \rho_0\, W'(\rho_0)$.

A different way to determine energy dissipation:

This equation can directly be integrated and due to the matching conditions we obtain

$$\left[\!\left[\left(W(\rho_0) + \frac{\rho_0}{2}|\mathbf{u}_0|^2 \right) (u_{\nu 0} - w_{\nu 0}) + p(\rho_0) u_{\nu 0} \right]\!\right] = 0$$

$$\left[\!\left[\left(W(\rho_0) + p(\rho_0) + \frac{\rho_0}{2}|\mathbf{u}_0 - \mathbf{w}_0|^2 \right) (u_{\nu 0} - w_{\nu 0}) \right]\!\right] = 0$$

$$\left[\!\left[\left(\rho_0 W'(\rho_0) + \frac{\rho_0}{2}|\mathbf{u}_0 - \mathbf{w}_0|^2 \right) (u_{\nu 0} - w_{\nu 0}) \right]\!\right] = 0$$

$$\left[\!\left[\rho_0 (u_{\nu 0} - w_{\nu 0}) \left(W'(\rho_0) + \frac{1}{2}|\mathbf{u}_0 - \mathbf{w}_0|^2 \right) \right]\!\right] = 0$$

i.e. there is no zeroth order entropy dissipation at the interface.

A different way to determine energy dissipation:

Similarly applying the coordinate change and inner expansion to the entropy equality and gathering the $\mathcal{O}(\varepsilon^0)$ terms, gives an equation, which (using a lot of technicalities) can be integrated and implies

$$
\begin{aligned}
0 \geqslant &-\varepsilon \overbrace{(\lambda + 2\mu)}^{>0} j_0^2 \int_{-\infty}^{\infty} \left(\left(\frac{1}{R_0} \right)_z \right)^2 dz + \mathcal{O}(\varepsilon^2) \\
&= \frac{\partial W_\Gamma}{\partial \tau} - (\gamma_\Gamma - W_\Gamma)\left(\mathrm{div}_\Gamma((w_\theta)_\varepsilon) - \kappa_\varepsilon (w_\nu)_\varepsilon\right) \\
&\quad + \left[\!\left[\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon) \left(W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2} \right) \right]\!\right],
\end{aligned}
$$

with

$$
W_\Gamma = \varepsilon \int_0^\infty \left(W(R_0) - W(\rho_0^+) + \frac{1}{2}\frac{j_0^2}{R_0} - \frac{1}{2}\frac{j_0^2}{\rho_0^+} + \frac{\gamma}{2} R_{0,z}^2 \right) dz + \varepsilon \int_{-\infty}^0 \dots
$$

$$
\gamma_\Gamma = \varepsilon \int_0^\infty \left(\frac{j_0^2}{R_0} - \frac{j_0^2}{\rho_0^+} + \gamma R_{0,z}^2 \right) dz + \varepsilon \int_{-\infty}^0 \dots
$$

Summary

- For NSK an entropy inequality for the SI limit can be derived using the energy equality for the diffuse model.
- In fact it even gives the dissipation rate.
- The kinetic relation has to be in agreement with the entropy inequality.
- If the kinetic relation is derived via a sharp interface limit, the kinetic relation has to be compatible with the condition from the energy equality.
- For mixtures without viscosity the situation is more involved -> Talk of Clemens Guhlke.