An overdetermined problem with non constant boundary condition

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joint work (in progress) with
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The original Serrin's problem

J. Serrin, *A symmetry problem in potential theory*
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for some $r > 0$ (up to translations).

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Technique: MOVING PLANE METHOD!

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Of course Ω is no more a ball!
Even though... it is known that if $|\nabla u| \sim$ Const, then $\Omega \sim$ a ball.
See for instance [Aftalion-Busca-Reichel, Adv. Diff. Eq. 1999] and [Brandolini-Nitsch-S.-Trombetti, JDE 2008].

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Problem close to [Gustafsson-Shahgholian, J. Reine Angew. Math., 1996].
They study $\quad -\Delta u = f \quad$ where f is a function (or a measure) *whose positive part f_+ has compact support*.
This makes a real difference as the radial case shows.

The torsional rigidity

For any bounded open set Ω we denote by u_Ω the solution of of the *torsion* problem (u_Ω is sometimes called the *stress* function of Ω)

$$\begin{cases} -\Delta u_\Omega = 1 & \text{in } \Omega \\ u_\Omega = 0 & \text{on } \partial\Omega \end{cases} \tag{0.2}$$

or, in its weak form

$$u_\Omega \in H_0^1(\Omega), \ \ \forall\, v \in H_0^1(\Omega), \int_\Omega \nabla u_\Omega \nabla v = \int_\Omega u_\Omega\, v. \tag{0.3}$$

u_Ω is characterized also as

$$u_\Omega = \operatorname{argmin}\{G_\Omega(v), v \in H_0^1(\Omega)\} \text{ where}$$

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The *Torsional Rigidity* of Ω

$$\tau(\Omega) = -2G_\Omega(u_\Omega) = \int_\Omega u_\Omega dx = \int_\Omega |\nabla u_\Omega|^2 dx\,.$$

A shape optimization (and localization) problem

Problem

Maximize $\tau(\Omega)$ *with the constraint* $\int_\Omega g(x)^2 dx \leq 1$.

It is a variant of the famous Saint-Venant's problem (to maximize torsonial rigidity among sets with given measure), connected to the Serrin's problem. Here we have a not uniform density, driven by the function g^2.

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EQUIVALENTLY: Define

$$J(\Omega) = -\frac{1}{2}\tau(\Omega) = -\frac{1}{2}\int_\Omega |\nabla u_\Omega|^2 \, dx \tag{0.5}$$

and

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$$(SOPb) \quad \min\{J(\Omega)\ :\ \phi(\Omega) \leq 1\}\,.$$

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Strong A

$$\begin{cases} g \text{ Hölder continuous,} \\ \alpha\text{-homogeneous, i.e. } g(tx) = t^\alpha g(x) \text{ for every } t > 0, \text{ for some } 1 \neq \alpha > 0, \\ g > 0 \text{ outside } 0. \end{cases}$$

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Notice that, by homogeneity, g is completely determined by one of its level sets, say $G_1 = \{x \in \mathbb{R}^n \, : \, g(x) \leq 1\}$ and the degree of homogeneity α.
In fact, to solve the shape optimization problem (SOPb), it is sufficient to assume the following:

Weak A

$g \in C(\mathbb{R}^n)$ and $\lim_{|x| \to \infty} g(x) = +\infty$.

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Then, thanks to the homogeneity of g, we have

$$|\nabla u_{t\Omega}(x)| = t^{1-\alpha}\lambda\, g(x)$$

and the overdetermined problem (0.1) is solved by $t\Omega$ where
$t = \lambda^{1/(\alpha-1)}$ if $\alpha \neq 1$.

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The case $\alpha = 1$ is really special. As we can see explicitly in the radially symmetric case, it is possible to have no solution or an infinite number of solutions. Indeed, let $g(x) = a|x|$: as it is easily proved by Schwarz symmetrization, the solution has to be a ball. Now, looking for a ball of radius R solving (0.1) is equivalent to solve $g(R) = R/N$ (because $u_{B_R} = (R^2 - |x|^2)/2N$) and the result follows according to the value of a.

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A possible different approach is to consider the following penalized minimization problem (instead that the constrained one):

$$\min\{F(\Omega) = J(\Omega) + \frac{1}{2}\phi(\Omega)\}$$

as in [Gustafsson-Shahgholian 1996] or [Alt-Caffarelli 1981].
But an inspection of the radial case again shows that F may be unbounded and

$$\inf F(\Omega) = -\infty\,.$$

In particulat this happens when g is α-homogeneous with $\alpha < 1$.

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3. If g is α-homogeneous, then ϕ is homogeneous of degree $n+2\alpha$, i.e.

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Preliminary observations

4. By assumption *(Weak A)*, it follows that there exist a ball B_R such that $g \geq 1$ in $\mathbb{R}^n \setminus B_R$. Then the constraint $\phi(\Omega) \leq 1$ implies an uniform bound for the measures of the admissible sets:

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5. In turn the latter implies a lower bound for $J(\Omega)$; indeed, the solution of the Saint-Venant's problem tells us that the ball maximizes torsional rigidity among sets with given measure, then

$$J(\Omega) \geq J(B_r) = -\frac{1}{2}\tau(B_r) \quad \text{where } r = (R^n + \omega_n^{-1})^{1/n}\,.$$

Existence and regularity

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Then we are in the compactness situation and from any minimizing sequence we can extract a subsequence converging to some Ω, up to translations, that is there exists a minimizing sequence Ω_n and a sequence of translations $y_n \in \mathbb{R}^n$, such that $\Omega_n + y_n$ γ-converge to $\tilde{\Omega}$.

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Thanks again to the behaviour of g at ∞, we can argue as in [Bucur-Buttazzo-Velichkov, 2011] to get that y_n is bounded (then it converges to som y_0 up to a subsequence) and to finally obtain a minimizing sequence Ω_n converging (with no translation) to $\Omega = \tilde{\Omega} - y_0$.

Regularity

Assume *(Weak A)* and $g > 0$ outside 0.

Regularity (outside 0) goes as in [Briancon-Hayouni-Pierre 2005], [Briancon, 2004], [Gustafsson-Shahgholian, 1996], [Alt-Caffarelli, 1981].

Then we have $C^{1,\beta}$ regularity in dimension 2 (in $\mathbb{R}^n$ with $n \geq 3$ we have the same for the reduced boundary, which coincides with $\partial\Omega$ up to a set of zero $\mathbb{H}^{n-1}$ measure).

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$0 \in \Omega$

If we assume *(Strong A)* with $\alpha > 1$, we can prove that 0 is in the interior of Ω and in dimension 2 we have $C^{1,\beta}$ regularity for the whole $\partial\Omega$.

Uniqueness

Proposition

Assume *(Strong A)* with $\alpha > 1$.
Then there exists at most one bounded solution Ω of the overdetermined problem (0.1).

Proof. By contradiction $\Omega_1 \neq \Omega_2$.

$t = sup\{s \,:\, s\Omega_1 \subseteq \Omega_2\}$, $0 < t < 1$

$t\Omega_1 \subset \Omega_2$, $\bar{x} \in \partial\Omega_2 \cap \partial(t\Omega_1) \neq \emptyset$

$u_{t\Omega_1}(x) = t^2 u_{\Omega_1}(x/t)$,

$|\nabla u_{t\Omega_1}(\bar{x})| = t|\nabla u_{\Omega_1}(x/t)| = tg(\bar{x}/t)$

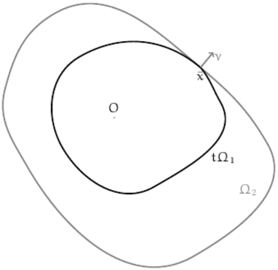

FIGURE 2. $t\Omega_1 \subseteq \Omega_2$ with $\bar{x} \in \partial(t\Omega_1) \cap \partial\Omega_2$

By comparison $u_{\Omega_2} \geq u_{t\Omega_1}$ in $t\overline{\Omega}_1$, while $u_{\Omega_2}(\bar{x}) = u_{t\Omega_1}\bar{x}$, then

$$g(\bar{x}) = |\nabla u_{\Omega_2}(\bar{x})| \geq |\nabla u_{t\Omega_1}(\bar{x})| = t^{1-\alpha} g(\bar{x})$$

which is impossible if $\alpha > 1$ since $t < 1$.

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Lemma

If $x \in \partial\Omega^* \setminus \partial\Omega$, then

$$|\nabla u_{\Omega^*}(x)| \geq \left((1-\lambda)\sqrt{|\nabla u(x_0)|} + \lambda\sqrt{|\nabla u(x_1)|}\right)^2,$$

where $x_0, x_1 \in \partial\Omega$ and $\lambda \in (0,1)$ are such that $x = (1-\lambda)x_0 + \lambda x_1$.

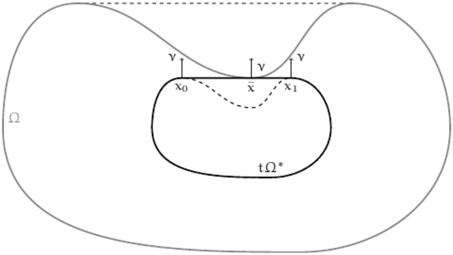

FIGURE 4. $t = \sup\{s \in [0,1]\ s\Omega^* \subseteq \Omega\}$.

Relation between Ω and G_1

If G_1 is a ball, that is if g is radial, it is easily seen (by a Schwarz rearrangement) that Ω must be a ball.

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Question:

Is there any relation in general between the optimal shape Ω and the shape dictated by g?

Relation between Ω and G_1

If G_1 is a ball, that is if g is radial, it is easily seen (by a Schwarz rearrangement) that Ω must be a ball.

Then the solution Ω has the same shape of the level sets of g.

Notice that Ω is a level set of g if and only if radial situation (by Serrin)

Question:

Is there any relation in general between the optimal shape Ω and the shape dictated by g?

To give an answer, let us introduce some notation.
Denote by v the stress function of G_1, i.e.

$$\begin{cases} -\Delta v = 1 & \text{in } G_1 = \{x \,:\, g(x) < 1\}\,, \\ u = 0 & \text{on } \partial G_1 = \{x \,:\, g(x) = 1\}\,. \end{cases}$$

Set

$$A = \min_{\partial G_1} |\nabla v|\,, \qquad B = \max_{\partial G_1} |\nabla v|\,.$$

Notice that $A \leq B$ and in fact $A < B$ unless G_1 is a ball (again Serrin).

Relation between Ω and G_1

Theorem

Assume *(Strong A)* with $\alpha > 1$. Then

$$A^{1/(\alpha-1)} G_1 \subseteq \Omega \subseteq B^{1/(\alpha-1)} G_1 \, .$$

$$G_r \subseteq \Omega \subseteq G_s$$

$$G_r = \{g \leq r\}$$
$$r = A^{\alpha/(\alpha-1)}$$

$$G_s = \{g \leq s\}$$
$$s = B^{\alpha/(\alpha-1)}$$

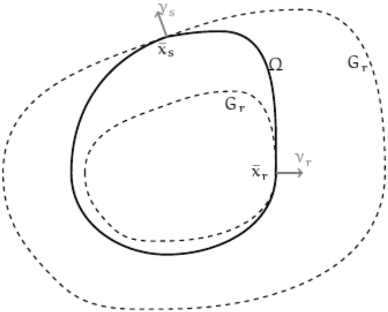

FIGURE 1. $G_r \subseteq \Omega \subseteq G_s$

Stability of the radial symmetry

We can use the previous theorem to investigate the stability of the radial symmetry.

The idea is very simple: g is close to be radial if G_1 is close to be a ball; then the previous result tells us that Ω is close to be a ball, provided we can give some bound about A and B.

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Then we will assume G_1 close to a ball in a C^2-sense and we prove that Ω is close to a ball.

Stability

Let $\alpha > 1$ and G_1 be a C^2 convex set and assume that there exists $R > 0$ and (a small enough) $\epsilon > 0$ such that

$$R - \epsilon \leq r_1(x) \leq \cdots \leq r_{n-1}(x) \leq R + \epsilon \quad \text{for every } x \in \partial G_1 \,, \tag{0.8}$$

where $r_1(x), \ldots, r_n(x)$ denote the principal radii of curvature of ∂G_1 at x. Then

$$d_H(\Omega, B) \leq \frac{\alpha}{\alpha - 1} \left(\frac{R}{n} \right)^{1/(\alpha-1)} \epsilon \,.$$

where B denotes the ball centered at 0 with radius $r = R^{\alpha/(\alpha-1)}$.