Global solvability of a problem governing the motion of two incompressible capillary fluids

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Problem background

• The problem on the motion of two fluids was first investigated by J. Hadamard and by V. Rybczynski in 1911. They obtained an analytic expression for the solution of the Stokes system corresponding to the axisymmetric fall of a spherical drop in another fluid with constant velocity.

• Stationary motion of two fluids with unknown closed interface between them was studied by V. Ja. Rivkind ('77-'84) and by J. Bemelmans ('81). Unfortunately, both investigations have inaccuracies. Later, V. A. So-lonnikov gave a correct proof of the solvability of the problem governing stationary fall (or rise) of a drop in liquid medium ('96, '99).

• V. Ja. Rivkind and N. Fridman proved the existence of a solution of a nonlinear nonstationary problem with a given fixed interface between the fluids ('73).

• In complete statement, the problem on the motion of two fluids with and without surface tension taken into account on the unknown closed interface was first studied by I. V. Denisova. Using V. A. Solonnikov's technique developed for a single fluid in vacuum, she proved a local (in time) existence theorem for the problem in Sobolev–Slobodetskiĭ functional spaces ('89-'90).

• Global (in time) solvability of the problem was studied by N. Tanaka who also used V. A. Solonnikov's technique. But it is necessary to note that the demonstrations of existence theorem in his paper of 1993 were not convincing.

• Results of Y. Giga and Sh. Takahashi ('94-'95) and of A. Nouri, F. Poupaud and Y. Demay ('93-'97) concern the existence of global weak solutions for the Stokes and Navier-Stokes equations describing the motion of two (or several) immiscible fluids with different densities and viscosities, in the absence of surface tension.

• We proved a local (in time) existence theorem for the problem with surface tension in Hölder spaces ('91-'95).

• I. V. Denisova obtained global solvability of the problem on two-phase fluid motion without taking surface tension into account (2007). This result was proved in Hölder spaces.

• Now new researchers began to analyze the problem for two-phase flow. J. Prüss, G. Simonett found conditions for solution analyticity for the nonlinear problem with an initial interface close to a half-plane ('09). H. Abels studied the situation when only weak solutions to the problem existed, he estimated the Hausdorff measure of the interface ('07).

Abstract

• We deal with the motion of two incompressible fluids in a container.

• The liquids are separated by an unknown interface on which the surface tension is taken into account.

• Global existence theorem is proved in anisotropic Hölder classes for a small initial velocity and for a initial configuration of the inner drop close to a ball $\{|x| \leqslant R_0\}$ with drop volume [1].

• We show that fluid velocity $\downarrow$ exponentially as $t \to \infty$ and the interface between the liquids $\to$ a sphere $S^2_{R_0}(h_\infty) = \{|x - h_\infty| = R_0\}$ with a center h_∞ close to 0.

• The proof is based on a local existence theorem in Hölder spaces [2] and on an exponential estimate of L_2-norms of local solutions.

• We follow to V. A. Solonnikov's scheme for proving global solvability of a problem on the motion of a single drop with free surface [3].

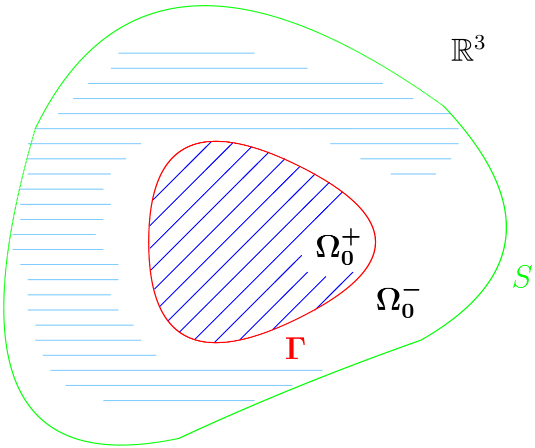
1 Denisova I. V., Solonnikov V. A., Zap. nauchn. semin. POMI **397** (2011), 20–52. (English transl. in *J. Math. Sci.* to appear).

2 Den., Sol. Algebra i Analiz, **7**(1995), no.5, 101–142 (Russian) (English transl. in St. Petersburg Math. J., **7** (1996), no.5, 755–786).

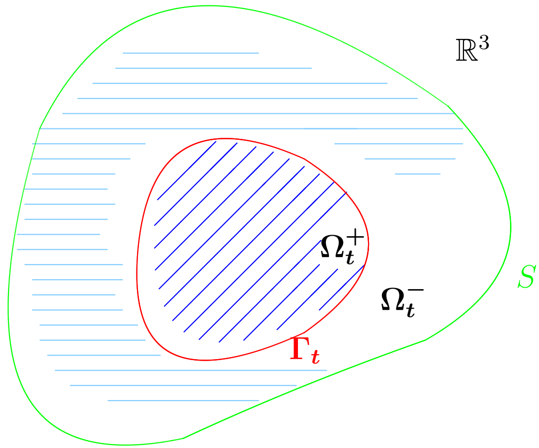
3 Solonnikov V. A., *Lectures Notes in Maths.*, **1812**, (2003), 123–175.

1 Statement of the problem

$t = 0$: Let a fluid with the viscosity $\nu^+ > 0$ and the density $\rho^+ > 0$ occupy a bounded domain $\Omega_0^+ \subset \mathbb{R}^3$; we denote $\partial\Omega_0^+$ by Γ. And let a fluid with the viscosity $\nu^- > 0$ and the density $\rho^- > 0$ fill a domain Ω_0^- surrounding Ω_0^+. The boundary $S \equiv \partial(\Omega_0^+ \cup \Gamma \cup \Omega_0^-)$ is a given closed surface, $S \cap \Gamma = \emptyset$.



$t = 0$



$t > 0$

For $\boldsymbol{t > 0}$ **:** $\Gamma_t = \partial\Omega_t^+$–?, velocity vector field $\boldsymbol{v}(x,t) = (v_1, v_2, v_3)$–? and the function p–? that is the deviation from the hydrostatic pressure P_0, which satisfy the following initial–boundary value problem :

$$\begin{aligned}
&\mathcal{D}_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu^{\pm}\nabla^2\boldsymbol{v} + \frac{1}{\rho^{\pm}}\nabla p = 0, \ \ \nabla\cdot\boldsymbol{v} = 0 \quad \text{in} \quad \Omega_t^{\pm},\ t>0,\\
&\boldsymbol{v}|_{t=0} = \boldsymbol{v}_0 \quad \text{in} \quad \Omega_0^-\cup\Omega_0^+, \qquad \boldsymbol{v}|_S = 0,\\
&[\boldsymbol{v}]|_{\Gamma_t} \equiv \lim_{\substack{x\to x_0\in\Gamma_t,\\ x\in\Omega_t^+}} \boldsymbol{v}(x) - \lim_{\substack{x\to x_0\in\Gamma_t,\\ x\in\Omega_t^-}} \boldsymbol{v}(x) = 0, \quad [\mathbb{T}(\boldsymbol{v},p)\boldsymbol{n}]|_{\Gamma_t} = \sigma H\boldsymbol{n} \ \text{ on } \Gamma_t.
\end{aligned} \tag{1}$$

Here $\mathcal{D}_t = \frac{\partial}{\partial t}$, $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$, $\nu^{\pm}, \rho^{\pm}$ are the step functions of viscosity and density, respectively, $\boldsymbol{v}_0$ is the initial distribution of the velocity, $\mathbb{T}(\boldsymbol{v},p)$ is the stress tensor with the elements $T_{ik} = -\delta_i^k p + \mu^{\pm}(\partial v_i/\partial x_k + \partial v_k/\partial x_i)$, $i,k = 1,2,3$; $\mu^{\pm} = \nu^{\pm}\rho^{\pm}$, δ_i^k is the Kronecker symbol, $\sigma \geqslant 0$ is surface tension coefficient, $\boldsymbol{n}$ is the outward normal to Ω_t^+, $H(x,t)$ is twice the mean curvature of Γ_t ($H < 0$ at the points where Γ_t is convex towards Ω_t^-). We suppose that a Cartesian coordinate system $\{x\}$ is introduced in $\mathbb{R}^3$. The centered dot denotes the Cartesian scalar product.

We imply the summation from 1 to 3 with respect to repeated indexes. We mark the vectors and the vector spaces by boldface letters.

We suppose the inner domain Ω_0^+ is close to a ball of its volume. $\Rightarrow$ We introduce a new pressure function $p_1 = p$ in Ω^+ and $p_1 = p + \sigma\frac{2}{R_0}$ in Ω^-. Then the last boundary condition in (1) changes:

$$\begin{aligned}
&\mathcal{D}_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu^{\pm}\nabla^2\boldsymbol{v} + \frac{1}{\rho^{\pm}}\nabla p_1 = 0, \quad \nabla\cdot\boldsymbol{v} = 0 \quad \text{in} \quad \Omega_t^{\pm},\ t>0,\\
&\boldsymbol{v}|_{t=0} = \boldsymbol{v}_0 \quad \text{in} \quad \Omega_0^-\cup\Omega_0^+, \qquad \boldsymbol{v}|_S = 0,\\
&[\boldsymbol{v}]|_{\Gamma_t} = 0, \quad [\mathbb{T}(\boldsymbol{v},p_1)\boldsymbol{n}]|_{\Gamma_t} = \sigma\Big(H + \frac{2}{R_0}\Big)\boldsymbol{n} \ \text{ on } \Gamma_t.
\end{aligned} \tag{2}$$

We assume the liquids to be immiscible $\Rightarrow$ A condition excluding the mass transportation through Γ_t. $\Leftrightarrow$ Γ_t consists of the points $x(\xi,t)$ whose radius vector $\boldsymbol{x}(\xi,t)$ is a solution of the Cauchy problem

$$\mathcal{D}_t\boldsymbol{x} = \boldsymbol{v}(x(\xi,t),t), \quad \boldsymbol{x}(\xi,0) = \boldsymbol{\xi}, \quad \xi\in\Gamma, \quad t>0, \tag{3}$$

where $\Gamma \equiv \Gamma_0 = \partial\Omega_0^+$ is a surface given at the initial moment. Hence, $\Omega_t^{\pm} = \{x = x(\xi,t) | \xi\in\Omega_0^{\pm}\}$.

Condition (3) completes system (2).

Transformation into the Lagrangian coordinates

We transform **the Eulerian coordinates** $\{x\}$ **into the Lagrangian ones** $\{\xi\}$ by the formula

$$\boldsymbol{x}(\xi,t) = \boldsymbol{\xi} + \int_0^t \boldsymbol{u}(\xi,\tau)d\tau \equiv \boldsymbol{X}_{\boldsymbol{u}}(\xi,t), \tag{4}$$

where $\boldsymbol{u}(\xi,t)$ is velocity vector field in the Lagrangian coordinates.

We apply **the well known relation for twice the mean curvature:**

$$H\boldsymbol{n} = \Delta(t)\boldsymbol{x} = \Delta(t)\boldsymbol{X}_{\boldsymbol{u}},$$

where $\Delta(t)$ is the Beltrami–Laplace operator on Γ_t.

We separate **the boundary condition for the stress tensor in (2) onto the tangential and normal components.** Let $\boldsymbol{n}_0$ be the outward normal to Γ.

2 Problem in the Lagrangian coordinates

$\Rightarrow$ We arrive at the problem for $\boldsymbol{u}$ and $q = p_1(X_{\boldsymbol{u}}, t)$ with the given interface Γ. If $\boldsymbol{n} \cdot \boldsymbol{n}_0 > 0$ this system is equivalent to the following one:

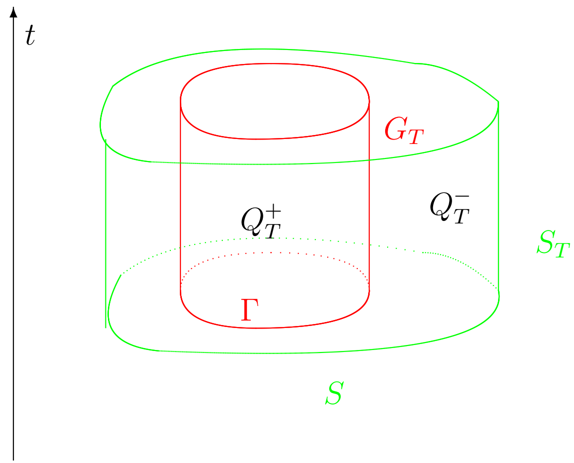
$$\begin{aligned}
&\mathcal{D}_t \boldsymbol{u} - \nu^{\pm} \nabla_{\boldsymbol{u}}^2 \boldsymbol{u} + \frac{1}{\rho^{\pm}} \nabla_{\boldsymbol{u}} q = 0, \quad \nabla_{\boldsymbol{u}} \cdot \boldsymbol{u} = 0 \quad \text{in } Q_T^{\pm} = \Omega^{\pm} \times (0, T), \quad \Omega^{\pm} \equiv \Omega_0^{\pm}, \\
&\boldsymbol{u}|_{t=0} = \boldsymbol{v}_0 \quad \text{in } \Omega^- \cup \Omega^+, \\
&[\boldsymbol{u}]|_{G_T} = 0, \quad \boldsymbol{u}|_S = 0, \quad [\Pi_0 \Pi \mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u}) \boldsymbol{n}]|_{G_T} = 0, \\
&[\boldsymbol{n}_0 \cdot \mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u}, q) \boldsymbol{n}]|_{G_T} - \sigma \boldsymbol{n}_0 \cdot \Delta(t) X_{\boldsymbol{u}}|_{G_T} = \sigma \frac{2}{R_0} \boldsymbol{n}_0 \cdot \boldsymbol{n}.
\end{aligned} \tag{5}$$

The notation: $\nabla_{\boldsymbol{u}} = \mathbb{A}\nabla$, $\mathbb{A}$ is the matrix of cofactors A_{ij} to the elements $a_{ij}(\xi, t) = \delta_i^j + \int_0^t \frac{\partial u_i}{\partial \xi_j} \mathrm{d}t'$ of the Jacobian matrix of (4), the vector $\boldsymbol{n}$ is connected with $\boldsymbol{n}_0$ by the relation $\boldsymbol{n} = \mathbb{A}\boldsymbol{n}_0/|\mathbb{A}\boldsymbol{n}_0|$; $\Pi\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{\omega})$, $\Pi_0\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{n}_0(\boldsymbol{n}_0 \cdot \boldsymbol{\omega})$ are the projections of a vector $\boldsymbol{\omega}$ onto the tangent planes to Γ_t and to Γ, respectively. The tensor $\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{w}, q)$ has the elements

$$(\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{w}, q))_{ij} = -\delta_j^i q + \mu^{\pm}(A_{jk}\partial w_i/\partial \xi_k + A_{ik}\partial w_j/\partial \xi_k),$$

$H_0(\xi) = \boldsymbol{n}_0 \cdot \Delta(0)\boldsymbol{\xi}$ is twice the mean curvature of Γ.

We denote: $Q_T^{\pm} \equiv \Omega_0^{\pm} \times (0,T)$, $D_T \equiv Q_T^+ \cup Q_T^-$, $G_T \equiv \Gamma \times (0,T)$, $S_T \equiv S \times (0,T)$.



We remind the definition of Hölder functional spaces. Let Ω be a domain in $\mathbb{R}^n$, $n \in \mathbb{N}$; for $T > 0$ we put $\Omega_T = \Omega \times (0, T)$; finally, let $\alpha \in (0, 1)$. By $C^{\alpha,\alpha/2}(\Omega_T)$ we denote the set of functions f in Ω_T having norm

$$|f|^{(\alpha,\alpha/2)}_{\Omega_T} = |f|_{\Omega_T} + \langle f\rangle^{(\alpha,\alpha/2)}_{\Omega_T} ,$$

where

$$|f|_{\Omega_T} = \sup_{t\in(0,T)} \sup_{x\in\Omega} |f(x,t)|, \qquad \langle f\rangle^{(\alpha,\alpha/2)}_{\Omega_T} = \langle f\rangle^{(\alpha)}_{x,\Omega_T} + \langle f\rangle^{(\alpha/2)}_{t,\Omega_T},$$

$$\begin{aligned} \langle f\rangle^{(\alpha)}_{x,\Omega_T} &= \sup_{t\in(0,T)} \sup_{x,y\in\Omega} |f(x,t) - f(y,t)||x-y|^{-\alpha}, \\ \langle f\rangle^{(\mu)}_{t,\Omega_T} &= \sup_{x\in\Omega} \sup_{t,\tau\in(0,T)} |f(x,t) - f(x,\tau)||t-\tau|^{-\mu}, \quad \mu \in (0,1). \end{aligned}$$

We introduce the following notation:

$$\begin{aligned} \mathcal{D}^{\boldsymbol{r}}_x &= \partial^{|\boldsymbol{r}|}/\partial x_1^{r_1} \dots \partial x_n^{r_n}, \quad \boldsymbol{r} = (r_1, \dots r_n), \quad r_i \geqslant 0, \ |\boldsymbol{r}| = r_1 + \dots + r_n, \\ \mathcal{D}^s_t &= \partial^s/\partial t^s, \quad s \in \mathbb{N} \cup \{0\}. \end{aligned}$$

Let $k \in \mathbb{N}$. By definition, the space $C^{k+\alpha,(k+\alpha)/2}(\Omega_T)$ consists of functions f with finite norm

$$|f|_{\Omega_T}^{(k+\alpha,\frac{k+\alpha}{2})} = \sum_{|\boldsymbol{r}|+2s\leqslant k} |\mathcal{D}_x^{\boldsymbol{r}}\mathcal{D}_t^s f|_{\Omega_T} + \langle f\rangle_{\Omega_T}^{(k+\alpha,\frac{k+\alpha}{2})},$$

where

$$\langle f\rangle_{\Omega_T}^{(k+\alpha,\frac{k+\alpha}{2})} = \sum_{|\boldsymbol{r}|+2s=k} \langle \mathcal{D}_x^{\boldsymbol{r}}\mathcal{D}_t^s f\rangle_{\Omega_T}^{(\alpha,\frac{\alpha}{2})} + \sum_{|\boldsymbol{r}|+2s=k-1} \langle \mathcal{D}_x^{\boldsymbol{r}}\mathcal{D}_t^s f\rangle_{t,\Omega_T}^{(\frac{1+\alpha}{2})}.$$

The symbol $C_0^{k+\alpha,\frac{k+\alpha}{2}}(\Omega_T)$ denotes the subspace of $C^{k+\alpha,\frac{k+\alpha}{2}}(\Omega_T)$ whose elements f has the property: $\mathcal{D}_t^i f\big|_{t=0} = 0,\ i = 0, \ldots, \left[\frac{k+\alpha}{2}\right]$.

We define $C^{k+\alpha}(\Omega)$, $k \in \mathbb{N}\cup\{0\}$, as the space of functions $f(x)$, $x \in \Omega$, with the norm

$$|f|_{\Omega}^{(k+\alpha)} = \sum_{|\boldsymbol{r}|\leqslant k} |\mathcal{D}_x^{\boldsymbol{r}} f|_{\Omega} + \langle f\rangle_{\Omega}^{(k+\alpha)}.$$

Here

$$\langle f\rangle_{\Omega}^{(k+\alpha)} = \sum_{|\boldsymbol{r}|=k} \langle \mathcal{D}_x^{\boldsymbol{r}} f\rangle_{\Omega}^{(\alpha)} = \sum_{|\boldsymbol{r}|=k} \sup_{x,y\in\Omega} |\mathcal{D}_x^{\boldsymbol{r}} f(x) - \mathcal{D}_y^{\boldsymbol{r}} f(y)||x-y|^{-\alpha}.$$

We also need the following semi-norm with $\alpha, \gamma \in (0,1)$:

$$\boldsymbol{|} f \boldsymbol{|}_{\Omega_T}^{(\gamma,1+\alpha)} = \boldsymbol{\langle} f \boldsymbol{\rangle}_{\Omega_T}^{(\gamma,1+\alpha)} + \langle f\rangle_{t,\Omega_T}^{(\frac{1+\alpha-\gamma}{2})},$$

where

$$\langle f\rangle_{\Omega_T}^{(\gamma,1+\alpha)} = \max_{t,\tau\in(0,T)} \max_{x,y\in\Omega} \frac{|f(x,t)-f(y,t)-f(x,\tau)+f(y,\tau)|}{|x-y|^{\gamma}|t-\tau|^{(1+\alpha-\gamma)/2}}.$$

It is known the estimate

$$\langle f\rangle_{\Omega_T}^{(\gamma,1+\alpha)} \leqslant c_1 \langle f\rangle_{\Omega_T}^{(1+\alpha,\frac{1+\alpha}{2})}.$$

We consider that $f\in C^{(\gamma,1+\alpha)}(\Omega_T)$ if $|f|_{\Omega_T}+|f|_{\Omega_T}^{(\gamma,1+\alpha)}<\infty.$

Finally, if a function f has finite norm

$$|f|_{\Omega_T}^{(\gamma,\mu)} \equiv \langle f\rangle_{x,\Omega_T}^{(\gamma)} + |f|_{t,\Omega_T}^{(\mu)}, \qquad \gamma\in(0,1), \quad \mu\in[0,1),$$

where

$$|f|_{t,\Omega_T}^{(\mu)} = \begin{cases} |f|_{\Omega_T} + \langle f\rangle_{t,\Omega_T}^{(\mu)} & \text{if } \mu>0, \\ |f|_{\Omega_T} & \text{if } \mu=0, \end{cases}$$

then it belongs to the Hölder space $C^{\gamma,\mu}(\Omega_T)$.

Let us set $\cup Q_T^{\pm} = Q_T^- \cup Q_T^+$ and

$$|f|_{Q_T^{\pm}}^{(k+\alpha)} = |f|_{Q_T^-}^{(k+\alpha)} + |f|_{Q_T^+}^{(k+\alpha)},$$

$$|f|_{\cup\Omega^{\pm}}^{(k+\alpha)} = |f|_{\Omega^-}^{(k+\alpha)} + |f|_{\Omega^+}^{k+\alpha)}.$$

Theorem 2.1. *(Global existence theorem) Suppose that* $\Gamma \in C^{3+\alpha}$, $\boldsymbol{v}_0 \in \boldsymbol{C}^{2+\alpha}(\Omega_0^- \cup \Omega_0^+)$, $\sigma \in C^{3+\alpha}(\mathbb{R}_+)$, $\sigma > 0$, $S \in C^{2+\alpha}$ *with some* $\alpha \in (0,1)$*. Assume also the compatibility conditions hold:*

$$\begin{aligned} &\nabla \cdot \boldsymbol{v}_0 = 0, \quad \boldsymbol{v}_0|_S = 0, \quad [\boldsymbol{v}_0]|_\Gamma = 0, \\ &[\mu^\pm \Pi_0 \mathbb{S}(\boldsymbol{v}_0)\boldsymbol{n}_0]\big|_\Gamma = 0, \qquad [\Pi_0(\nu^\pm \nabla^2 \boldsymbol{v}_0 - \frac{1}{\rho^\pm}\nabla q_0)]\big|_\Gamma = 0, \\ &\Big(\Pi_S(\nu^- \nabla^2 \boldsymbol{v}_0 - \frac{1}{\rho^-}\nabla q_0)\Big)\Big|_S = 0, \end{aligned} \tag{6}$$

where $q_0(\xi) \equiv p_1(\xi, 0)$ *is a solution of the diffraction problem*

$$\begin{aligned} &\frac{1}{\rho^\pm}\nabla^2 q_0(\xi) = -\nabla \cdot \mathcal{D}_t \mathbb{B}^*\big|_{t=0} \boldsymbol{v}_0(\xi), \qquad \xi \in \Omega_0^- \cup \Omega_0^+, \\ &[q_0]|_\Gamma = \Big[2\mu^\pm \frac{\partial \boldsymbol{v}_0}{\partial \boldsymbol{n}_0} \cdot \boldsymbol{n}_0\Big]\Big|_\Gamma - \sigma\Big(H_0(\xi) + \frac{2}{R_0}\Big), \qquad \xi \in \Gamma, \\ &\Big[\frac{1}{\rho^\pm}\frac{\partial q_0}{\partial \boldsymbol{n}_0}\Big]\Big|_\Gamma = [\nu^\pm \boldsymbol{n}_0 \cdot \nabla^2 \boldsymbol{v}_0]\big|_\Gamma, \qquad \frac{1}{\rho^-}\frac{\partial q_0}{\partial \boldsymbol{n}_S}\Big|_S = \nu^- \boldsymbol{n}_S \cdot \nabla^2 \boldsymbol{v}_0\big|_S. \end{aligned} \tag{7}$$

Here $\mathbb{B} = \mathbb{A} - \mathbb{I}$, $\mathbb{I}$ *is the identity matrix,* $\mathbb{B}^*$ *is the transpose to* $\mathbb{B}$, $\boldsymbol{n}_S$ *is the outward normal to* S, $\Pi_S \boldsymbol{\omega} \equiv \boldsymbol{\omega} - \boldsymbol{n}_S(\boldsymbol{n}_S \cdot \boldsymbol{\omega})$, $\frac{\partial}{\partial \boldsymbol{n}_0} = \boldsymbol{n}_0 \cdot \nabla$, $\frac{\partial}{\partial \boldsymbol{n}_S} = \boldsymbol{n}_S \cdot \nabla$, $H_0(\xi) = \boldsymbol{n}_0 \cdot \Delta(0)\xi\big|_\Gamma$ *is twice the mean* *curvature of* Γ.

Moreover, let for $t = 0$ Γ *is given by the equation* $|x| = R\Big(\frac{x}{|x|}, 0\Big)$ *on the unit sphere* S_1 *and the initial data are small enough, i.e.*

$$|\boldsymbol{v}_0|^{(2+\alpha)}_{\cup\Omega_0^\pm} + |r_0|^{(3+\alpha)}_{S_1} \leqslant \varepsilon \ll 1, \tag{8}$$

where $r_0(x) = R(x, 0) - R_0$, R_0 *is the radius of the ball* B_{R_0}: $|\Omega_0^+| = 4\pi R_0^3/3$.

$\Rightarrow$ *Problem* (2), (3) *is uniquely solvable on the whole positive half-axis* $t > 0$, *and the solution* $(\boldsymbol{v}, p_1)$ *has the properties:* $\boldsymbol{v} \in \boldsymbol{C}^{2+\alpha,1+\alpha/2}$, $p_1 \in C^{(\gamma,1+\alpha)}$, $\nabla p_1 \in \boldsymbol{C}^{\alpha,\alpha/2}$, *the function* p_1 *being defined up to a bounded time dependent function. The interface* Γ_t *is given for every* t *by a function of* $C^{3+\alpha}$: $|x - h(t)| = R\Big(\frac{x-h}{|x-h|}, t\Big)$, *it tends to a sphere of the radius* R_0 *with the center in some point* h_∞. *It means that for arbitrary* $t_0 \in (0, \infty)$ *the solution* $(\boldsymbol{u}, q)$ *and its derivative in Lagrangian coordinates are in the corresponding functional spaces over* $D_{(t_0,t_0+\tau)} \equiv \cup Q^\pm_{(t_0,t_0+\tau)}$ *for a sufficiently small time interval* $(t_0, t_0 + \tau)$. *In addition, it holds the estimate*

$$\begin{aligned} N_{(t_0,t_0+\tau)}[\boldsymbol{v}, p_1, r] &\equiv |\boldsymbol{u}|^{(2+\alpha,1+\alpha/2)}_{D_{(t_0,t_0+\tau)}} + |\nabla q|^{(\alpha,\alpha/2)}_{D_{(t_0,t_0+\tau)}} + |q|^{(\gamma,1+\alpha)}_{D_{(t_0,t_0+\tau)}} + \sup_{t\in(t_0,t_0+\tau)} |r(\cdot, t)|^{(3+\alpha)}_{S_1} \\ &\leqslant c e^{-bt_0}\big\{|\boldsymbol{v}_0|^{(2+\alpha)}_{\cup\Omega_0^\pm} + |r_0|^{(3+\alpha)}_{S_1}\big\}, \end{aligned} \tag{9}$$

where $r(\omega, t) = R(\omega, t) - R_0$.

3 Auxiliary propositions

Theorem 3.1. ***(Local existence theorem.)*** *Let the hypotheses of Theorem* 2.1 *be satisfied.*
$\Rightarrow$ *For an arbitrary* $T > 2$ *there exists a such* $\varepsilon(T)$ *that problem* (5) *has a unique solution* $(\boldsymbol{u}, q)$ *with the properties:* $\boldsymbol{u} \in \boldsymbol{C}^{2+\alpha,1+\alpha/2}(D_{ST})$, $q \in C^{(\gamma,1+\alpha)}(D_{ST})$, $\nabla q \in \boldsymbol{C}^{\alpha,\alpha/2}(D_{ST})$ *provided that*

$$|\boldsymbol{v}_0|^{(2+\alpha)}_{\cup\Omega_0^\pm} + |H_0 + \frac{2}{R_0}|^{(1+\alpha)}_\Gamma \leqslant \varepsilon(T). \tag{10}$$

The interface Γ_t *is a surface of* $C^{3+\alpha}$*–class and it is given by*

$$x = \boldsymbol{h}(t) + y + \boldsymbol{N}(y) r\Big(\frac{y}{|y|}, t\Big), \quad y \in S_{R_0} = \{|y| = R_0\},$$

where $N(y) = y/|y|$, $\boldsymbol{h}(t) = \frac{1}{|\Omega_t^+|} \int\limits_0^t \int\limits_{\Omega_t^+} \boldsymbol{u}(\xi, \tau)\, \mathrm{d}\xi\, \mathrm{d}\tau$ *is the barycenter of* Ω_t^+. *Moreover, there holds the estimate*

$$\begin{aligned} N_{(0,T)}[\boldsymbol{v}, p_1, r] &\leqslant c\Big\{|\boldsymbol{v}_0|^{(2+\alpha)}_{\cup\Omega_0^\pm} + |H_0 + \frac{2}{R_0}|^{(1+\alpha)}_\Gamma\Big\} \\ &\leqslant c\Big\{|\boldsymbol{v}_0|^{(2+\alpha)}_{\cup\Omega_0^\pm} + |r_0|^{(3+\alpha)}_{S_1}\Big\}. \end{aligned} \tag{11}$$

Proposition 3.1. *Solution of problem* (5) *satisfies the inequality*

$$N_{(t-1,t)}[\boldsymbol{v}, p_1, r] \leqslant c\Big\{ \int_{t-2}^{t} \|\boldsymbol{v}(\cdot,\tau)\|_{L_2(\Omega)}\,\mathrm{d}\tau + \int_{t-2}^{t} \|r(\cdot,\tau)\|_{W_2^1(S_1)}\,\mathrm{d}\tau \Big\}, \tag{12}$$

$\forall t \in (2, T]$. *In addition, for* $\forall t \in (0, T]$, *the estimate*

$$\|\boldsymbol{v}(\cdot,t)\|_{L_2(\Omega)} + \|r(\cdot,t)\|_{W_2^1(S_1)} \leqslant c_1\,\mathrm{e}^{-bt}\Big\{ \|\boldsymbol{v}_0\|^2_{L_2(\Omega)} + \|r_0\|^2_{W_2^1(S_1)} \Big\} \tag{13}$$

holds. Here $\Omega = \overline{\Omega_t^+} \cup \Omega_t^-$.

Corollary 3.1. *For an arbitrary* $t > 2$ *we have*

$$N_{(t-1,t)}[\boldsymbol{v}, p_1, r] \leqslant c\,\mathrm{e}^{-bt}\Big\{ \|\boldsymbol{v}_0\|^2_{L_2(\Omega)} + \|r_0\|^2_{W_2^1(S_1)} \Big\}. \tag{14}$$

Corollary 3.2. *The coordinates of the barycenter of* Ω_t^+ *satisfy the inequality*

$$|\boldsymbol{h}(t)| \leqslant \frac{1}{|\Omega_t^+|^{1/2}} \int_0^t \|\boldsymbol{v}(\cdot,\tau)\|_{L_2(\Omega)}\,\mathrm{d}\tau \leqslant c\Big\{ \|\boldsymbol{v}_0\|_{L_2(\Omega)} + \|r_0\|_{W_2^1(S_1)} \Big\}, \qquad t \in (0, T].$$

3.1 Main steps of the proof of Prop. 3.1

First we prove *the exponential estimate*. Multiplying the first equation in (2) by $\rho^{\pm}\boldsymbol{v}$ and integrating by parts over $\Omega_t^- \cup \Omega_t^+$, obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{\pm}\boldsymbol{v}\|_{2,\Omega}^2 + \frac{\mu^+}{2}\|\mathbb{S}(\boldsymbol{v})\|_{2,\Omega_t^+}^2 + \frac{\mu^-}{2}\|\mathbb{S}(\boldsymbol{v})\|_{2,\Omega_t^-}^2 = \sigma\int_{\Gamma_t}\big(H + \frac{2}{R_0}\big)\boldsymbol{n}\cdot\boldsymbol{v}\,\mathrm{d}\Gamma.$$

Use the formula

$$H\boldsymbol{n} = \Delta_{\Gamma_t}\boldsymbol{x},$$

here Δ_{Γ_t} is the Beltrami–Laplace operator on Γ_t, and

$$\sigma\int_{\Gamma_t}\big(H + \frac{2}{R_0}\big)\boldsymbol{n}\cdot\boldsymbol{v}\,\mathrm{d}\Gamma = \sigma\int_{\Gamma_t}\boldsymbol{v}\cdot\Delta(t)\boldsymbol{x}\,\mathrm{d}\Gamma = -\sigma\frac{\mathrm{d}}{\mathrm{d}t}|\Gamma_t|.$$

$\Rightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{1}{2}\|\rho^{\pm}\boldsymbol{v}\|_{2,\Omega}^2 + \sigma\big(|\Gamma_t| - 4\pi R_0^2\big)\Big\} + \frac{\mu^+}{2}\|\mathbb{S}(\boldsymbol{v})\|_{2,\Omega_t^+}^2 + \frac{\mu^-}{2}\|\mathbb{S}(\boldsymbol{v})\|_{2,\Omega_t^-}^2 = 0.$$

Since $\boldsymbol{v}\big|_S = 0$, the Korn inequality holds:

$$\|\boldsymbol{v}\|_{W_2^1(\Omega)} \leqslant c_0 \|\mathbb{S}(\boldsymbol{v})\|_{2,\Omega}. \tag{15}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{1}{2}\|\rho^{\pm}\boldsymbol{v}\|^2_{2,\Omega} + \sigma\big(|\Gamma_t| - 4\pi R_0^2\big)\Big\} + c_1\|\boldsymbol{v}\|^2_{W_2^1(\Omega)} \leqslant 0. \tag{16}$$

Multiply the first equation in problem (2) by $\rho^{\pm}\boldsymbol{W}$, where $\boldsymbol{W}$ is a solenoidal smooth function satisfying special estimates, and integrate by parts:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^{\pm}\boldsymbol{v}\cdot\boldsymbol{W}\,\mathrm{d}x \;+\; \int_\Omega \rho^{\pm}\boldsymbol{v}\cdot(\mathcal{D}_t\boldsymbol{W} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{W})dx + \int_\Omega \frac{\mu^{\pm}}{2}\mathbb{S}(\boldsymbol{v}) : \mathbb{S}(\boldsymbol{W})\,\mathrm{d}x$$
$$-\sigma\int_{\Gamma_t}\big(H + \frac{2}{R_0}\big)\boldsymbol{n}\cdot\boldsymbol{W}\,\mathrm{d}\Gamma = 0. \tag{17}$$

Add equality (17) multiplied by small γ to inequality (16). $\Rightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{1}{2}\|\rho^{\pm}\boldsymbol{v}\|^2_{2,\Omega} \;+\; \gamma\int_\Omega \rho^{\pm}\boldsymbol{v}\cdot\boldsymbol{W}\,\mathrm{d}x + \sigma\big(|\Gamma_t| - 4\pi R_0^2\big)\Big\} + \gamma\int_\Omega \frac{\mu^{\pm}}{2}\mathbb{S}(\boldsymbol{v}) : \mathbb{S}(\boldsymbol{W})\,\mathrm{d}x$$
$$+c_1\|\boldsymbol{v}\|^2_{W_2^1(\Omega)} \;-\; \gamma\int_\Omega \rho^{\pm}\boldsymbol{v}\cdot(\mathcal{D}_t\boldsymbol{W} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{W})\,\mathrm{d}x + \gamma\sigma\int_{S_{R_0}}(H + \frac{2}{R_0})\widetilde{r}\,\mathrm{d}S_{R_0} \leqslant 0.$$

We set the generalized energy as follows

$$\mathcal{E}(t) = \frac{1}{2}\|\rho^{\pm}\boldsymbol{v}\|^2_{2,\Omega} + \gamma \int_{\Omega} \rho^{\pm}\boldsymbol{v}\cdot\boldsymbol{W}\,\mathrm{d}x + \sigma\{|\Gamma_t| - 4\pi R_0^2\}.$$

We also put

$$\begin{aligned}\mathcal{E}_1(t) = c_1\|\boldsymbol{v}\|^2_{W_2^1(\Omega)} \quad &- \quad \gamma\int_{\Omega}\rho^{\pm}\boldsymbol{v}\cdot(\mathcal{D}_t\boldsymbol{W} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{W})\,\mathrm{d}x \\ &+ \quad \gamma\int_{\Omega}\frac{\mu^{\pm}}{2}\mathbb{S}(\boldsymbol{v}):\mathbb{S}(\boldsymbol{W})\,\mathrm{d}x - \gamma\sigma\int_{S_{R_0}}(H + \frac{2}{R_0})\widetilde{r}\,\mathrm{d}S_{R_0}.\end{aligned}$$

We construct such a function $\boldsymbol{W}$ that

$$\mathcal{E}_1(t) \geqslant b\mathcal{E}(t).$$

$\Rightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + b\mathcal{E}(t) \leqslant 0$$

from which it follows the exponential estimate due to the Gronwall lemma.

4 The main steps of the proof of global existence Theorem 2.1

4.1 First step

By local existence theorem (Th. 3.1), $\exists$ a solution $(\mathbf{v}, p_1)$ on $(0, T_0]$. The magnitude of T_0 depends on the norms of the initial data $\sim \varepsilon$.

$$N_{(0,T_0)}[\mathbf{v}, p_1] \leqslant c\Big(|\mathbf{v}_0|^{(2+\alpha)}_{\cup\Omega_0^\pm} + |H_0 + \frac{2}{R_0}|^{(1+\alpha)}_{\Gamma}\Big) \leqslant c\varepsilon. \tag{18}$$

We can take $T > 2$ if ε is sufficiently small.

Remind: $\Omega = \overline{\Omega_t^+} \cup \Omega_t^-$, $|\Omega|$ is the measure of Ω. Prop. 3.1 (the exp. estimate) $\Rightarrow$ for $\forall t_0 \in (2, T]$

$$\begin{aligned} N_{(t_0-1,t_0)}[\mathbf{v}, p_1, r] &\leqslant c_6\,\mathrm{e}^{-bt_0}\Big\{\|\mathbf{v}_0\|_{2,\Omega} + \|r_0\|_{W_2^1(S_1)}\Big\} \\ &\leqslant c_6\,\mathrm{e}^{-bt_0}\Big(|\Omega|^{\frac{1}{2}}|\mathbf{v}_0|^{(2+\alpha)}_{\cup\Omega_0^\pm} + 2\pi^{\frac{1}{2}}|r_0|^{(3+\alpha)}_{S_1}\Big). \end{aligned}$$

$$\Rightarrow |\boldsymbol{u}(\cdot, T)|^{(2+\alpha)}_{\cup\Omega_T^\pm} + |r(\cdot, T)|^{(3+\alpha)}_{S_1} \leqslant \frac{1}{2}\{|\boldsymbol{v}_0|^{(2+\alpha)}_{\cup\Omega_0^\pm} + |r_0|^{(3+\alpha)}_{S_1}\},$$
$$|\boldsymbol{h}(t)| \leqslant a, \qquad t \leqslant T,$$

where $a = \frac{c_1}{|\Omega_0^+|^{1/2}} \int\limits_0^\infty \mathrm{e}^{-bt}\,\mathrm{d}t\Big\{\|\boldsymbol{v}_0\|_{L_2(\Omega)} + \|r_0\|_{W_2^1(S_1)}\Big\} \leqslant c_2\varepsilon.$

So, the barycenter of Ω_t^+ tends to a point h_∞ which is moved by not more than the distance a, it being small as small ε. Hence, we can extend the solution on $(T, 2T)$.

4.2 Second step

We repeat our arguments. Then we have

$$|\boldsymbol{u}(\cdot, 2T)|_{\cup\Omega_{2T}^{\pm}}^{(2+\alpha)} + |r(\cdot, 2T)|_{S_1}^{(3+\alpha)} \leqslant \frac{1}{2}\{|\boldsymbol{u}(\cdot, T)|_{\cup\Omega_T^{\pm}}^{(2+\alpha)} + |r(\cdot, T)|_{S_1}^{(3+\alpha)}\} \leqslant \frac{1}{4}\{|\boldsymbol{v}_0|_{\cup\Omega_0^{\pm}}^{(2+\alpha)} + |r_0|_{S_1}^{(3+\alpha)}\},$$

and for the barycenter of the internal fluid

$$\begin{aligned} |\boldsymbol{h}(t)| &\leqslant |\boldsymbol{h}(T)| + \frac{1}{|\Omega_0^+|^{1/2}} \int_T^t \|\boldsymbol{u}(\xi^{(1)}, \tau)\|_{L_2(\Omega)}\, \mathrm{d}\tau \\ &\leqslant \frac{1}{|\Omega_0^+|^{1/2}} \Big\{ \int_0^T \|\boldsymbol{u}(\xi, \tau)\|_{L_2(\Omega)}\, \mathrm{d}\tau + \int_T^t \|\boldsymbol{u}(\xi^{(1)}, \tau)\|_{L_2(\Omega)}\, \mathrm{d}\tau \Big\} \leqslant a, \end{aligned}$$

$t \in (T, 2T)$. And so on.

4.3 Transform into the Lagrangian coordinates

Now we transform the Eulerian coordinates $\{x\}$ into the Lagrangian ones $\{\xi^{(1)}\}$ by a new formula

$$\boldsymbol{X} = \boldsymbol{\xi}^{(1)} + \int_T^t \boldsymbol{u}(\xi^{(1)}, \tau)\,\mathrm{d}\tau, \qquad \xi^{(1)} \in \cup\Omega_T^{\pm}, \quad t \in (T, 2T), \tag{19}$$

but really formula (1) is the same as earlier one:

$$\boldsymbol{X}(\xi, t) = \boldsymbol{\xi} + \int_0^T \boldsymbol{u}(\xi, \tau) d\tau + \int_T^t \boldsymbol{u}(\xi, \tau) d\tau, \qquad \xi \in \cup\Omega_0^{\pm}, \quad t \in (T, 2T).$$

The same remark applies to the barycenter of the internal fluid, since the volume of the fluid is conserved:

$$\mathbf{h}(t) = \mathbf{h}(T_0) + \int_{T_0}^{t} \frac{1}{|\Omega_t^+|} \int_{\Omega_t^+} \mathbf{v}(x, \tau)\,\mathrm{d}x\,\mathrm{d}\tau = \frac{3}{4\pi R_0^3} \int_0^t \int_{\Omega_t^+} \mathbf{v}(x, \tau)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Thus, we can extend the solution on an infinite interval.

4.4 Limiting position of the barycenter of the internal fluid

The limiting position of the barycenter is estimated from the inequality

$$
\begin{aligned}
|h_\infty| \leqslant a \quad &\leqslant \quad c_9 \Big\{ \|\mathbf{v}_0\|_{2,\Omega} + \|r_0\|_{W_2^1(S_1)} \Big\} \\
&\leqslant \quad c_{10} \Big(|\mathbf{v}_0|^{(2+\alpha)}_{\cup\Omega_0^\pm} + |r_0|^{(3+\alpha)}_{S_1} \Big) \leqslant c_{10}\varepsilon.
\end{aligned} \tag{20}
$$

Hence the initial distance between the surfaces Γ and S should be strictly larger than $c_{10}\Big(|\mathbf{v}_0|^{(2+\alpha)}_{\cup\Omega_0^\pm} + |r_0|^{(3+\alpha)}_{S_1}\Big) + \delta_1 R_0$, in order that the intersection of these surfaces in the future would be excluded.

Finally, using estimates obtained we prove solution uniqueness.

Theorem 2.1 is proved.

Conclusions.

• Unsteady motion of a drop in another incompressible fluid bounded by a rigid surface is considered.

• Global existence theorem for the problem is proved in Hölder classes of functions provided that the initial velocity of the liquids has small norm and the initial configuration of the drop is close to a ball with center in drop's barycenter.

• It is shown that velocity vector field decays exponentially as $t \to \infty$ and the interface between the liquids tends to a sphere $\{|x - h_\infty| = R_0\}$ with a center $h_\infty \in \mathbb{R}^3$ close to drop's barycenter.

• If initial data are small enough, the inner liquid will remain inside the other one during all the time.