Macroscopic electrodynamics of hard superconductors

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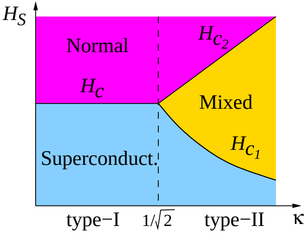
FBP2012, June 11, 2012

Outline

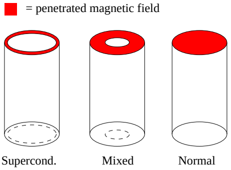
Type-II superconductors

The behavior (at $T < T_c$) of a superconducting sample in an external magnetic field $\vec{H}_S$ is characterized by the Ginzburg-Landau parameter κ of the material.



H_c: critical field (type-I)

H_{c_1}, H_{c_2}: critical fields (type-II)



- Superconducting phase: thin layer (20-50 nm); no magnetic field in the bulk of the superconductor.
- Mixed state: partial penetration in the bulk.
- Normal conducting phase: full penetration in the bulk.

Bean's model

Bean's model (C.P. Bean, 1962): critical state model for the description of macroscopic electrodynamics for type-II hard superconductors.
Main assumption: there exists a critical current J_c such that:

- $|\vec{J}| = J_c$ in the region penetrated by the magnetic field;
- $\vec{J} = 0$ otherwise.

Anisotropy of J_c, due to Cu-O planes, structure of defects, etc: exists $\Delta \subset \mathbb{R}^3$ compact convex containing a neighborhood of 0 s.t.

- $\vec{J} \in \partial\Delta$, in the region penetrated by the magnetic field;
- $\vec{J} = 0$ otherwise.

Macroscopic electrodynamics

PROBLEM: given a superconductor $Q \subset \mathbb{R}^3$ in an external field $\vec{H}_S(t)$, find the internal magnetic field $\vec{H}(x,t)$ and the electric field $\vec{E}(x,t)$.

- Faraday's law: $\operatorname{curl} \vec{E} = -\mu_0 \dfrac{\partial \vec{H}}{\partial t}$
- Ampère's law: $\vec{J} = \operatorname{curl} \vec{H}$
- (Modified) Ohm's law: $\vec{E} = \vec{E}(\vec{J})$

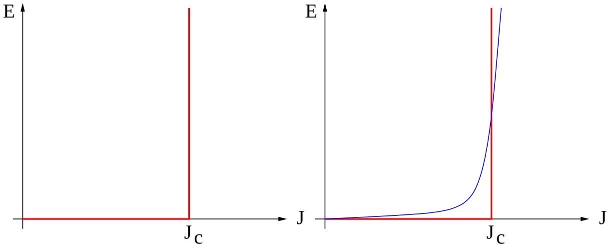
Examples of material laws (Ohm's law):

- isotropic conductor: $\vec{E}(\vec{J}) = r\,\vec{J}$, $r =$ resistivity
- anisotropic conductor: $\vec{E}(\vec{J}) = A\,\vec{J}$, $A =$ resistivity tensor
- isotropic power-law: $\vec{E}(\vec{J}) = c\left(\dfrac{|\vec{J}|}{J_c}\right)^p \vec{J}$

  ($\vec{E}$ and $\vec{J}$ have the same direction).

Problem: dependence $\vec{E} = \vec{E}(\vec{J})$ in the Bean's anisotropic model.

In the isotropic case, the constraint $|\vec{J}| \leq J_c$ can be described by a vertical $\vec{E}$–$\vec{J}$ relation:



...that can be approximated by a power-law relation

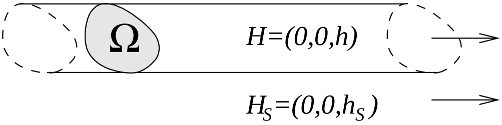
$$|\vec{E}(\vec{J})| = c \left(\frac{|\vec{J}|}{J_c} \right)^p .$$

The electric field is determined using the additional condition $\vec{E} || \vec{J}$.

Problem: dependence $\vec{E} = \vec{E}(\vec{J})$ in the Bean's anisotropic model.

- Start from an anisotropic power law approximation for the dissipation $\vec{E} \cdot \vec{J}$:
$$\vec{E}(\vec{J}) \cdot \vec{J} = \frac{c}{p} \left(\rho_\Delta(\vec{J})\right)^p$$
(ρ_Δ = gauge function of Δ).
- Deduce the dependence $\vec{E}(\vec{J}) = \frac{c}{p} \left(\rho_\Delta(\vec{J})\right)^{p-1} D\rho_\Delta(\vec{J})$.
- In the limit as $p \to \infty$: $\vec{E}(\vec{J}) \in \partial I_\Delta(\vec{J})$
$$\partial I_\Delta(\vec{J}) = \begin{cases} \{0\}, & \text{if } \vec{J} \in \text{interior of } \Delta, \\ \{\lambda\, D\rho_\Delta(\vec{J});\ \lambda \geq 0\}, & \text{if } \vec{J} \in \partial\Delta, \\ \emptyset, & \text{if } \vec{J} \notin \Delta \end{cases}$$
subdifferential of the indicator function of Δ.
$\Longrightarrow$ gives the constraint $\vec{J} \in \Delta$.

Cylindrical symmetry



- $Q = \Omega \times \mathbb{R}$, cylinder with cross-section $\Omega \subset \mathbb{R}^2$, smooth, simply connected;
- $\vec{H}_S(t) = (0, 0, h_S(t))$ directed along the axis of the cylinder.

$\Longrightarrow$ By symmetry: $\vec{H}(x, t) = (0, 0, h(x_1, x_2, t))$
$\Longrightarrow$ $\vec{J} = \text{curl}\, \vec{H} = (\partial_{x_2} h, -\partial_{x_1} h, 0)$

Remark: $\vec{J} \in \Delta \iff Dh \in K$, where $K \subset \mathbb{R}^2$ is the rotation of the section $z = 0$ of Δ.

Quasistatic evolution

Time discretization in $[0, T]$: $\delta t = T/n$, $t_i = i\delta t$,
$\vec{H}_i = (0, 0, h_i) = \vec{H}(t_i)$, $\vec{E}_i = \vec{E}(t_i)$.

Goal: obtain the variational formulation of the anisotropic Bean's model proposed by Badía - López using Γ-convergence of the power law approximation.

Power law for dissipation: $\vec{E}(\vec{J}) \cdot \vec{J} = \frac{c}{p} \left(\rho_\Delta(\vec{J})\right)^p$

Discretized Faraday's law: $\operatorname{curl} \vec{E}_{i+1} = -\mu_0 \frac{\vec{H}_{i+1} - \vec{H}_i}{\delta t}$
$\Longrightarrow$ admits the variational formulation

$$J_p(h) = \int_\Omega \frac{1}{p} [\rho(Dh)]^p + \frac{\mu_0}{2c\delta t}(h - h_i)^2, \qquad h \in h_s(t_{i+1}) + W_0^{1,p}(\Omega)$$

i.e., h_{i+1} is the unique minimum point of J_p in $h_s(t_{i+1}) + W_0^{1,p}(\Omega)$.
$\rho = \rho_K =$ gauge function of $K \subset \mathbb{R}^2$.

Convergence

Theorem (G.C. - A. Malusa)

$u_p \in h_s(t_{i+1}) + W_0^{1,p}(\Omega)$: unique minimum point of J_p, $p \geq 1$.
$h_{i+1} \in h_s(t_{i+1}) + W_0^{1,1}(\Omega)$: unique minimum point of

$$J(u) = \int_\Omega I_K(Du) + (u - h_i)^2 \,, \qquad u \in h_s(t_{i+1}) + W_0^{1,1}(\Omega).$$

Then, for every $q > 1$, (u_p) converges to h_{i+1} in weak-$W^{1,q}$.

Conclusion: the variational formulation of Bean's law is based on functional J. Given h_i, we have $h_{i+1} =$ unique minimum point of J.

Remark: variational formulation proposed by Badía-López (2002) starting from physical considerations.

Candidate solution

h_{i+1}: unique minimum point of

$$J(u) = \int_\Omega I_K(Du) + (u - h_i)^2 \,, \qquad u \in h_s(t_{i+1}) + W_0^{1,1}(\Omega).$$

Minkowski distance w.r.t. K: $d(x) = \min_{y \in \partial\Omega} \rho_K^0(x - y)$

($\rho_K^0 =$ polar of the gauge function of K)

$\Longrightarrow$ viscosity solution of $\rho(Du) = 1$ in Ω, $u = 0$ on $\partial\Omega$.

Minkowski distance w.r.t. $-K$:

$d^-(x) = \min_{y \in \partial\Omega} \rho_{-K}^0(x - y) = \min_{y \in \partial\Omega} \rho_K^0(y - x)$

$\Longrightarrow$ viscosity solution of $-\rho(Du) = -1$ in Ω, $u = 0$ on $\partial\Omega$.

Solution of the minimum problem:

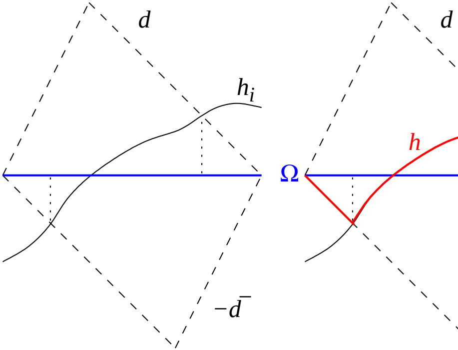
$$h_{i+1}(x) = \begin{cases} d(x) + h_s(t_{i+1}), & \text{if } x \in \Omega^+ = \{h_i > d\}, \\ -d^-(x) + h_s(t_{i+1}), & \text{if } x \in \Omega^- = \{h_i < -d^-\}, \\ h_i(x), & \text{if } x \in \Omega^0 = \Omega \setminus (\Omega^+ \cup \Omega^-). \end{cases}$$

1D heuristics

$K = [-1, 2]$

$J(h) = \int_\Omega |h - h_i|^2 + I_K(Dh)$

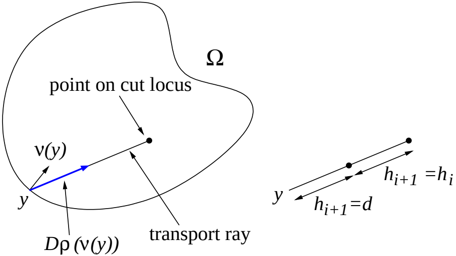
$h = 0$ on $\partial\Omega$



Decomposition of Ω in transport rays

Ω can be decomposed in transport rays (paths of minimal distance from the boundary):
two possible decompositions, one for d and one for d^-.

Example: $h_i(y) > 0$.



$\nu(y) =$ inward Euclidean normal of $\partial\Omega$ at y
$l(y) =$ length of the transport ray

$\Longrightarrow$ on each transport ray apply the 1D-heuristics.

Electric field

The variational formulation of the problem permits the computation of the main variable $\vec{H}$. Unfortunately, in the critical state model the electric field $\vec{E}$ cannot be computed using the current-voltage relation.

How to compute $\vec{E}$ for parallel geometry:

- Badía-López: compute $\vec{E}$ along paths of vortex penetration
- Barrett-Prigozhin: solve a dual variational problem for $\vec{E}$
- G.C.-Malusa (and also Cannarsa-Cardaliaguet): solve a mass transport problem of Monge-Kantorovich type

Electric field – mass transport approach

Theorem (Dual function)

∃ a non-negative continuous function v_i such that

$$-\operatorname{div}(v_i\, D\rho(Dh_i)) = h_{i-1} - h_i \qquad in\ \Omega.$$

Interpretation: $w_i = v_i/\delta t$ is the (discretized) dissipated power density, and $E_i = w_i\, D\rho(Dh_i)$ is the (discretized) electric field.
If $\Omega \in C^2$, v_i has an explicit representation in terms of the anisotropic principal curvatures of $\partial\Omega$ and the normal distance from cut locus.

Techniques developed in
G.C., Malusa: Trans. Amer. Math. Soc. 2007, Arch. Rational Mech. Anal. 2009, Calc. Var. 2012

Isotropic case (K = ball):
Cannarsa, Cardaliaguet, G.C., Giorgieri: Calc. Var. 2005

Selected references

- Badía, López, Phys. Rev. B 2002, J. Low Temp. Phys. 2003, J. Appl. Phys. 2004: anisotropic Bean's model
- Barrett, Prigozhin, Nonlinear Anal. 2000, Interf. Free Boundaries 2006, M3AS 2010: isotropic Bean's model, variational inequalities
- Brandt *et al.*, Phys. Rev. B 1996 and 2000: numerical and experimental data

Proof of the minimality of $h = h_{i+1}$

$$\partial I_K(\xi) = \begin{cases} \{0\}, & \text{if } \xi \in \operatorname{int} K \\ \{t\, D\rho(\xi);\ t \geq 0\}, & \text{if } \xi \in \partial K \\ \emptyset, & \text{if } \xi \notin K \end{cases}$$

$v_i \geq 0,\ v_i = 0$ in $\Omega^0 \Longrightarrow 2v_i(x)\, D\rho(Dh(x)) \in \partial I_K(Dh(x))$

For every $w \in h_s(t_{i+1}) + W_0^{1,1}(\Omega)$:

$I_K(Dw(x)) - I_K(Dh(x)) \geq 2v_i(x)\langle D\rho(Dh(x)),\, Dw(x) - Dh(x)\rangle$

$$\begin{aligned} J(w) - J(h) &\geq \int_\Omega 2v_i(x)\langle D\rho(Dh(x)),\, Dw(x) - Dh(x)\rangle \\ &\qquad + \int_\Omega (w - h_i)^2 - (h - h_i)^2 \\ \text{[Nec. cond.]} &= \int_\Omega 2(w-h)(h_i - h) + (w - h_i)^2 - (h - h_i)^2 \\ &= \int_\Omega (h - w)^2 \end{aligned}$$

Quasistatic evolution

- Start with $h(x,0) = h_0(x) \in \mathrm{Lip}_K(\Omega)$, $h_0 = h_S(0)$ on $\partial\Omega$.
- h_{i+1} = internal magnetic field at time t_{i+1}
  $\Longrightarrow$ solution of the minimization problem
  $$\min \left\{ \int_\Omega \frac{\mu_0}{2} |h - h_i|^2 + \delta t\, I_K(Dh);\ h \in h_S(t_{i+1}) + W_0^{1,1}(\Omega) \right\}$$
- By the existence and uniqueness theorem,
  $h_{i+1}(x) = \left[h_i(x) \vee (h_S(t_{i+1}) - d^-(x)) \right] \wedge (h_S(t_{i+1}) + d(x))$
- Explicit formula for monotone external field:
  1. h_S monotone increasing in $[0, T]$:
     $$h_i(x) = h_0(x) \vee (h_S(t_i) - d^-(x))$$
  2. h_S monotone decreasing in $[0, T]$:
     $$h_i(x) = h_0(x) \wedge (h_S(t_i) + d(x))$$

The limit $\delta t \to 0$

For $\delta t = T/n$, $n \in \mathbb{N}^+$, construct h_i as above and define $h^n(x,t) = h_i(x)$, for $t \in [t_i, t_{i+1})$

Assume monotone external field; as $n \to \infty$ ($\delta t \to 0$)

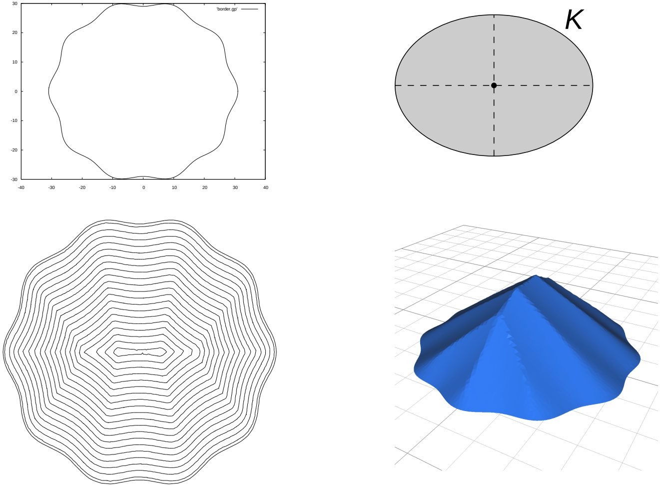
- h_S increasing: $h^n(x,t) \to h(x,t) = h_0(x) \vee (h_S(t) - d^-(x))$
- h_S decreasing: $h^n(x,t) \to h(x,t) = h_0(x) \wedge (h_S(t) + d(x))$

$\Longrightarrow$ the internal magnetic field can be explicitly computed if h_S is piecewise monotone.

In a similar way construct the approximated power dissipation $w^n(x,t)$, which converges pointwise to a function $w(x,t)$
$\Longrightarrow$ electric field: $\vec{E}(x,t) = w(x,t) D\rho(Dh(x,t))$.

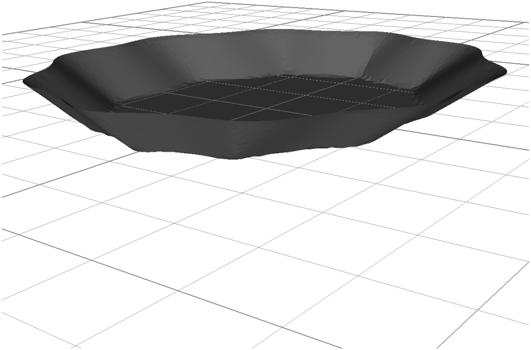
Convergence: $h^n \to h$ uniformly in $\overline{\Omega} \times [0,T]$
$w^n(t) \to w(t)$ in $L^p(\Omega)$, $p \geq 1$, uniformly in $[0,T]$.

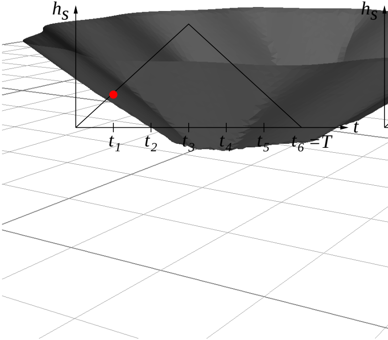
Example



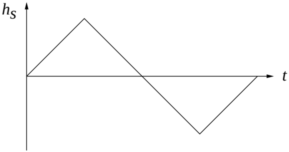
The section Ω, the constraints set K; Level sets and 3D-plot of the distance d.

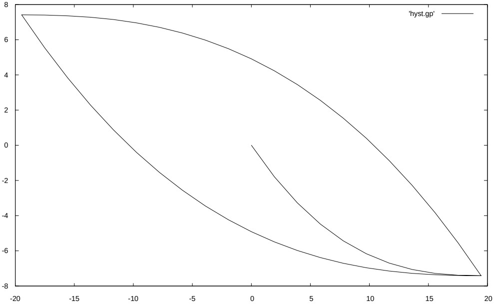
Example: plot of h





Hysteresis loop





Hysteresis loop: magnetization $\vec{M} = \langle \vec{H} \rangle - \vec{H}_S$ versus external field $\vec{H}_S$.

The electric field

$$\vec{E} \in \partial I_K(\vec{J}) \Longrightarrow \exists\ w(x,t) \geq 0 \text{ s.t. } \vec{E}(x,t) = w(x,t)\, D\rho(Dh(x,t)).$$

Meaning of $w(x,t)$: the power dissipation density of the sample is

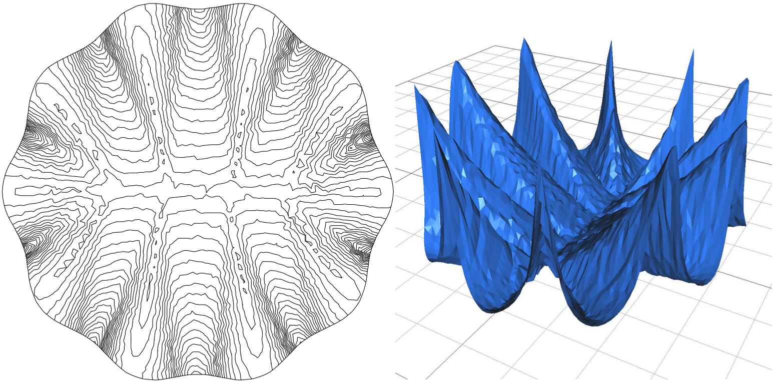
$$\begin{aligned}\vec{E}\cdot\vec{J} &= w(x,t)\,\langle D\rho(Dh(x,t)),\, Dh(x,t)\rangle \\ &= w(x,t)\,\rho(Dh(x,t)) = w(x,t)\end{aligned}$$

Construction of w:
in the discretized setting, from the necessary conditions we have unique functions v_i (with explicit integral representation) such that

$$-\operatorname{div}\left(\frac{v_{i+1}}{\delta t}\, D\rho(Dh_{i+1})\right) = -\frac{h_{i+1}-h_i}{\delta t}$$

Set $w_i = v_i/\delta t$ and $w^n(x,t) = w_i(x)$ for $t \in [t_i, t_{i+1})$.
Then $w^n \to w$, and $\vec{E} = w\, D\rho(Dh)$ satisfies Faraday's law.

Example: plot of w



Conclusion and outlook

What we have done...

- Strong mathematical justification of the anisotropic variational formulation of Bean's law suggested by Badía and López.
- Explicit form of both magnetic field and electric field inside the superconductor; explicit computation of the dissipated power density (very important for the stability analysis of the superconducting phase).

...and what remains to do:

- Nonhomogeneous samples (general Finsler metric instead of Minkowski); quasivariational approach by Barrett-Prigozhin 2010, Miranda-Rodrigues-Santos 2012, Rodrigues-Santos 2012.
- True 3D analysis (no cylindrical symmetry); samples with cavities